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SPACES OF θ-FAINTLY AND α-FAINTLY CONTINUOUS FUNCTIONS AND COMPOSITION OPERATORS

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ABSTRACT. This work is dedicated to the study of composition operator C_φ where φ is a continuous function. First, we characterize topologies on $C^\theta(X, Y)$, the space of θ-faintly continuous, to establish results for the composition operator C_φ,acting between spaces of θ-faintly continuous functions. Next, topologies on $C^\alpha(X, Y)$, the space of α-faintly continuous, are analyzed in the same way to obtain results again for the composition operator C_φ. In both cases, weak, strong and exponential topologies are investigated, obtaining inclusion properties and where the evaluation map is key.

1. INTRODUCTION

Continuity of functions has been studied through several variants such as θ-faintly continuity and α-faintly continuity due to Long and Herrington [9]. Also, on a space of functions, the topology of uniform convergence and, more generally, quasi-uniform convergence topologies have already been studied [3]. For their part, composition operators between spaces of functions take a leading role in functional analysis and this study has given rise to extensive literature [5, 11]. For X and Y, topological spaces, $\varphi : X \to X$, and $f : X \to Y$, continuous functions, the composition operator C_φ is defined as $C_\varphi(f) = f \circ \varphi$. Furthermore, the notion of θ-open set and θ-continuity was introduced by Velicko [16, 17]. Properties of faintly θs-continuous and faintly δs-continuous functions are included in [1] and this helps to understand how the variation of types of open sets considered affects the concept of continuity in order to shed some light on the behavior of composition operators in spaces of θ-continuous or α -continuous functions. We can also find works on connections between various types of continuity, such as quasi- and almost quasi-continuity [19]. Motivated by these developments, this work aims to the idea of finding topologies on spaces of continuous functions such that the evaluation map is continuous, which is one of the requirements in one of the main results in the characterization of Hurewicz spaces [20]. Additionally, it deals with the problem of topologizing spaces of θ-faintly continuous function and α-continuous functions so that the results on the composition operator C_φ

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are an extension of the existing results. For background on properties of continuous functions, see [12, 14, 10]. Types of continuity can also be found in [13], with a special focus on θ continuity and variations on α continuity have been studied in [6] providing results with multifunctions and characterization of composition operator acting between weighted (LF) and (PLB) spaces of continuous functions are given in [2]. Moreover, there exist developments of boundedness and mapping properties for composition (superposition) operators acting on generalized Morrey function spaces (a family of spaces that relate to many classical spaces of continuous functions)[7]. In addition, studies of the weak and weak-* topologies on function spaces: their barrelledness, Baire property, and sequential properties are treated in [8]. Some applications of α-open sets and θ-open can be viewed in [18, 15, 4]. Inspired by the aforementioned research, the aim is to study topologies on function spaces based on conditions on the evaluation map and the transpose map. The first two following sections study spaces of θ-faintly continuous functions and present results on composition operators between spaces of θ-faintly continuous functions. The next two sections include results on spaces of α-faintly continuous functions and composition operators between spaces of α-faintly continuous function spaces α-faintly continuous functions.

2. Spaces of θ-faintly continuous functions

Let (X, τ) be a topological space and consider $\mathcal{A} \subseteq X$. Then $\mathcal{A}$ is called θ-open if for each $x \in \mathcal{A}$ there exists an open set $\mathcal{B}$ such that $x \in \mathcal{B} \subseteq \overline{\mathcal{B}} \subseteq \mathcal{A}$. The set $\tau^\theta = \{U \subseteq X : U\ \theta - open\}$ defines a topology weaker than τ ($\tau^\theta \subseteq \tau$).

Let (X, τ) and (Y, τ') be topological spaces. A function $f : X \to Y$ is said to be:

- θ-faintly continuous function in $x \in X$ if for each θ-open V with $f(x) \in V$ there exists an open set U of X such that $x \in U$ and $f(U) \subseteq V$.
- θ-continuous function in $x \in X$ if for each θ-open V with $f(x) \in V$ there exists a θ-open set U of X such that $x \in U$ and $f(U) \subseteq V$.

We denote $C^\theta(X, Y)$ the set of θ-faintly continuous of X on Y. Notice that $C^\theta(X, Y) = C((X, \tau), (Y, \tau'^\theta))$. Let A be a topological space. The transpose map $\hat{T} : A \to C^\theta(X, Y)$ of a function $T : A \times X \to Y$ is given by $\hat{T}(a) = T_a$, being $T_a(x) = T(a, x)$. By δ_{XY} we mean the evaluation map:

$$\delta_{XY} : C^\theta(X, Y) \times X \to Y,$$
$$\delta_{XY}(f, x) = f(x).$$

Definition 2.1. A topology on $C^\theta(X, Y)$ is called:

- θ-weak if $T : A \times X \to Y$ θ-faintly continuous $\Longrightarrow \hat{T} : A \to C^\theta(X, Y)$ θ-faintly continuous,
- θ-strong if $\hat{T} : A \to C^\theta(X, Y)$ θ-faintly continuous $\Longrightarrow T : A \times X \to Y$ θ-faintly continuous,
- θ-exponential if it is θ-weak and θ-strong.

Proposition 2.2. *If a topology on $C^\theta(X, Y)$ is θ-strong then the evaluation map is a θ-faintly continuous function.*

Proof. Take $A = C^\theta(X,Y)$ and $T = \delta_{XY}$. So $\hat{T}(f) = \hat{\delta}_{XY}(f) = f$ and then $\hat{T}$ is the identity map on $C^\theta(X,Y)$. Since a θ-open set is an open set, it is followed that $\hat{T}$ is θ-faintly continuous. Now, by applying the hypothesis, $T = \delta_{XY}$ is θ-faintly continuous. □

Proposition 2.3. *If the evaluation map is a θ-continuous function, then the topology on $C^\theta(X,Y)$ is θ-strong.*

Proof. Assume that $\hat{T} : A \to C^\theta(X,Y)$ θ-faintly continuous, where $\hat{T}$ is the transpose map of T. Let $(a,x) \in A \times X$, $T(a,x) = y$, and V, θ-open in Y with $y \in V$. The θ-continuity of δ_{XY} in $(\hat{T}(a),x)$ implies the existence of a θ-open set $U_{\hat{T}(a)}$ in $C^\theta(X,Y)$ of $\hat{T}(a)$ and a θ-open set U_x in X (and so open) of x such that $\delta_{XY}(U_(\hat{T}(a)) \times U_x) \subseteq V$. For this $U_{\hat{T}(a)}$ we can find an open set U_a in A with $a \in U_a$ such as $\hat{T}(U_a) \subseteq U_{\hat{T}(a)}$. If $(a',x') \in U_a \times U_x$ then $T(a',x') = T_{a'}(x)$, $\hat{T}(a') = T_{a'} \in U_{\hat{T}(a)}$, $T_{a'}(x) = \delta_{XY}(T_{a'},x) \in V$. Therefore, $T(U_a \times U_x) \subseteq V$. □

Lemma 2.4. *The following properties are fulfilled:*

1 *Let τ_W and τ_S be a θ-weak and θ-strong topology, respectively. Then $(\tau_W)^\theta \subseteq \tau_S$*
2 *If τ is a topology on $C^\theta(X,Y)$ such that $\tau^\theta \subseteq (\tau_W)^\theta$ where τ_W is a θ-weak topology then τ is θ-weak topology.*
3 *If τ is a topology on $C^\theta(X,Y)$ such that $(\tau_S)^\theta \subseteq \tau^\theta$ where τ_S is a θ-strong topology then τ is θ-strong topology.*

Proof. 1. Applying proposition 2.2, it is obtained that $\delta_{XY} : (C^\theta(X,Y),\tau_S)\times X \to Y$ is θ-faintly continuous function. Since definition of τ_W it is followed that $\hat{\delta_{XY}} : (C^\theta(X,Y),\tau_S) \to (C^\theta(X,Y),\tau_W)$ is θ-faintly continuous. As $\hat{\delta_{XY}}$ is the identity map then $(\tau_W)^\theta \subseteq \tau_S$.

2. Let τ be a topology on $C^\theta(X,Y)$ such that $\tau^\theta \subseteq (\tau_W)^\theta$ and assume that $T : A \times X \to Y$ is θ-faintly continuous then $\hat{T} : A \to (C^\theta(X,Y),\tau_W)$, θ-faintly continuous. As $\tau^\theta \subseteq (\tau_W)^\theta$ then $id : (C^\theta(X,Y),\tau_W) \to (C^\theta(X,Y),\tau)$ is θ-continuous. So, $\hat{T} : A \to (C^\theta(X,Y),\tau)$, is θ-faintly continuous.

3. Let τ be a topology on $C^\theta(X,Y)$ such that $(\tau_S)^\theta \subseteq \tau^\theta$ and assume that $\hat{T} : A \to (C^\theta(X,Y),\tau)$, is θ-faintly continuous. As $(\tau_S)^\theta \subseteq \tau^\theta$ then $id : (C^\theta(X,Y),\tau) \to (C^\theta(X,Y),\tau_S)$ is θ-continuous then $\hat{T} : A \to (C^\theta(X,Y),\tau_S)$ is θ-faintly continuous. From definition, $T : A \times X \to Y$ is θ-faintly continuous. Therefore, τ is θ-strong. □

The θ-punctual convergence topology on $C^\theta(X,Y)$, $\tau_{\theta p}$ stands for the topology generated by the sets

$$W(x,V) = \{f : f(x) \in V\},$$

where $x \in X$ and V is a θ-open in Y. The θ-open compact topology on $C^\theta(X,Y)$, $\tau_{\theta co}$ stands for the topology generated by the sets

$$W(K,V) = \{f : f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and V is a θ-open in Y.

Notice that each $W(x,V)$ is θ-open, for all $x \in X$ and V is a θ-open in Y. Indeed, take $f \in W(x,V)$, as $f(x) \in V$, and V is θ-open there exists B open in Y such than $f(x) \in B \subseteq \overline{B} \subseteq V$. Therefore, $W(x,B) \subseteq W(x,\bar(B) \subseteq W(x,V)$. And $W(x,\overline{B})$ is closed since $(W(x,\overline{B}))^c = W(x,(\overline{B})^c)$ is open. Additionally, if $f \in W(K,V)$ then $f(K)$ is θ-compact.

Proposition 2.5. (1) *$\tau_\theta p$ is θ-weak.*
(2) $\tau_{\theta p} \subseteq \tau_{\theta co}$

Proof. 1.Assume $T : A \times X \to Y$ is θ-faintly continuous. Fix $a \in A$ and W θ-open in $(C^\theta(X,Y),\tau_{\theta p})$, with $\hat{T}(a) \in W$. Then it is possible to find $x \in X$ and V θ-open in Y such that $\hat{T}(a) \in W(x,V) \subseteq W$. So, $\hat{T}(a)(x) = T(a,x) \in V$. As T is θ-faintly continuous in (a,x), there exist U_a open in A and U_x open in X such that $(a,x) \in U_a \times U_x$ and $T(U_a \times U_x) \subseteq V$. Hence, $\hat{T}(U_a) \subseteq W(x,V) \subseteq W$.
2.Each $W(x,V) \in \tau_{\theta co}$, since $\{x\}$ is a compact set of X for all $x \in X$. □

A topological space Y is called θ-regular space if for each V is θ-open and each y in V there exists U, θ-open such that $y \in U \subseteq cl_\theta(U) \subseteq V$, that is (Y,τ'^θ) is a regular space.

In this case, each set $W(K,V)$ on $C^\theta(X,Y)$ is θ-open. Indeed, if $f \in W(K,V)$ then $f(K) \subseteq V$. For each $k \in K$ there exists $U_{f(k)}$ θ-open such that $f(k) \in U_(f(k)) \subseteq cl_\theta(U_{f(k)}) \subseteq V$. ($\overline{U_{f(k)}} \subseteq cl_\theta(U_{f(k)})$). So, $f(K) \subseteq \bigcup_{k\in K} U_{f(k)}$. Since f is θ-faintly continuous, $U_{f(k)}$ θ-open and K is compact we get $\{k_1,\cdots,k_n\}$ such that $f(K) \subseteq \bigcup_{j=1}^n U_{f(k_j)}$. Therefore $f \in W(K,U) \subseteq W(K,\overline{U}) \subseteq W(K,V)$, where $U = \bigcup_{j=1}^n U_{f(k_j)}$ and $\overline{U} = \bigcup_{j=1}^n(\overline{U_{f(k_j)}})$. If $g \in (W(K,\overline{U}))^c$ then there exists $k_0 \in K$ such that $g(k_0) \in \overline{U}^c$, that is $g \in W(k_0,\overline{U}^c)$ which is open in $\tau_{\theta co}$, now $W(k_0,\overline{U}^c) \subseteq (W(K,\overline{U}))^c$, thus $(W(K,\overline{U}))^c$ is open.

Proposition 2.6. *If X is compact topological space and Y is a θ-regular space, then $\tau_{\theta co}$ on $C^\theta(X,Y)$ is θ-strong.*

Proof. Assume $\hat{T} : A \to C^\theta(X,Y)$ is θ-faintly continuous. We must prove that $T : A\times X \to Y$ is θ-faintly continuous. Fix $(a,x) \in A\times X$, V is θ-open in Y with $T(a,x) \in V$, V is θ-open. As Y is θ regular, then we get $U_{T(a,x)}$ is θ-open such that $T(a,x) \in U_{T(a,x)} \subseteq cl_\theta(U_{T(a,x)}) \subseteq V$ and $\hat{T}(a)$ is θ-faintly continuous in x, then there exists U_x open such that $x \in U_x$, $\hat{T}(a)(U_x) \subseteq U_{T(a,x)}$. As X is compact then $W((\overline{U_x}),V)$ is θ-open that contains $\hat{T}(a)$ since $\hat{T}(a)(\overline{U_x}) \subseteq cl_\theta(\hat{T}(a)(U_x)) \subseteq cl_\theta(U_{T(a,x)}) \subseteq V$ By the hypothesis there exists U_a open in A containing a such that $\hat{T}(U_a) \subseteq W((\overline{U_x}),V)$. We can conclude that $T(U_a \times U_x) \subseteq V$. □

Corollary 2.7. *For X compact topological space and Y θ-regular space, the evaluation map $\delta_{XY} : C^\theta(X,Y) \times X \to Y$ is a θ-faintly continuous function for the topology $\tau_{\theta co}$ on $C^\theta(X,Y)$.*

Proposition 2.8. *If X is compact topological space and Y is a θ-regular space and $H \subseteq C^\theta(X,Y)$ considering $\tau_{\theta co}$ and H with the inherited topology $\tau_{\theta co_H}$ then $\delta_{XY} : H \times X \to Y$ is θ-faintly continuous function.*

Proof. The set $W(K,V)\bigcap H = \{f \in H : f(K) \subseteq V\}$ is a basis for $\tau_{\theta co_H}$.

The inclusion $\overline{T} : H \to C^{\theta}(X,Y)$ defined by $\hat{T}(f) = f$ is θ-faintly continuous. So, $\delta_{XY} = T : H \times X \to Y$ is θ-faintly continuous since $\tau_{\theta co}$ on $C^{\theta}(X,Y)$ is θ-strong. □

Example 2.9. Consider $C^{\theta}(X,Y)$ for $X = \overline{\mathbb{D}}$ the closed unit disk in the complex plane and $Y = \mathbb{C}$ the complex set endowed of usual topology which is a regular space. Here, the topology $\tau_{\theta co}$ is the open compact topology well known and this topology is θ-strong. Therefore, $\delta_{XY} : C^{\theta}(\overline{\mathbb{D}}, C) \times \overline{\mathbb{D}} \to C$ is a θ-faintly continuous function. If we consider the subspace $A(\overline{\mathbb{D}})$ of $C^{\theta}(\overline{\mathbb{D}}, Y)$ of continuous function on $\overline{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$ then $\delta_{XY} : A(\overline{\mathbb{D}}) \times X \to Y$ is θ-faintly continuous.

Example 2.10. Consider $X = Y = \mathbb{R}$, each endowed with usual topology. We define the convergence topology on θ-open with the base:

$\mathcal{B} = \{W(N,U) \subseteq C^{\theta}(X,Y) : N \subseteq X, N \text{ is } \theta - open, U \subseteq Y, Y \text{ is } open\}$, where

$$W(N,U) = \{f \in C^{\theta}(X,Y) : f(N) \subseteq U\}.$$

The evaluation map is θ-continuous with this topology on $C^{\theta}(X,Y)$.

3. Composition operators on spaces of θ-faintly continuous functions

Let $\varphi : X \to X$ be a continuous function, the composition operator $C_{\varphi} : C^{\theta}(X,Y) \to C^{\theta}(X,Y)$ is defined by: $C_{\varphi}(f) = f \circ \varphi$

Theorem 3.1. *Consider $C^{\theta}(X,Y)$ endowed of a θ-exponential topology τ. Then $C_{\varphi} : C^{\theta}(X,Y) \to C^{\theta}(X,Y)$ is θ-faintly continuous.*

Proof. The operator C_{φ} is well defined. Fix $f_0 \in C^{\theta}(X,Y)$ and $x_0 \in X$. As τ is θ- strong then $\delta_{XY} : C^{\theta}(X,Y) \times X \to Y$ is θ-faintly continuous in $(C_{\varphi}(f_0), x_0)$. So, for V, θ-open in Y, $\delta_{XY}(C_{\varphi}(f_0), x_0) \in V$ there exist open sets $U_{(C_{\varphi}(f_0))} \times U_{(}x_0)$ in $C^{\theta}(X,Y) \times X$ such that $\delta_{XY}(U_{C_{\varphi}(f_0)} \times U_{x_0}) \subseteq V$. Now, for $U_{C_{\varphi}(f_0)}$ is possible to find an open set U_{f_0} in $C^{\theta}(X,Y)$ such that $\varphi(U_{f_0}) \subseteq U_{C_{\varphi}(f_0)}$. Let $T : C^{\theta}(X,Y) \times X \to Y$ defined by $T(f,x) = C_{\varphi}(f)(x) = \delta_{XY}(C_{\varphi}(f_0), x_0)$. Hence, $T(U_{f_0} \times U_{x_0}) \subseteq V$. So, T is θ-faintly continuous. As τ is θ- weak then $\hat{T} : C^{\theta}(X,Y) \to C^{\theta}(X,Y)$ is θ-faintly continuous, $\hat{T}(f) = C_{\varphi}(f)$. □

Corollary 3.2. *Let H be a subset of $C^{\theta}(X,Y)$ endowed of an exponential topology τ. Then $C_{\varphi|H} : H \to C^{\theta}(X,Y)$ is θ-faintly continuous.*

4. Spaces of α-faintly continuous functions

Let (X,τ) be a topological space and consider $U \subseteq X$. Then A is called α-open if $A \subseteq int(cl(int(A)))$. The set $\tau^{\alpha} = \{U \subseteq X : U \ \alpha - open\}$ defines a topology stronger than τ) $(\tau) \subseteq \tau^{\alpha})$.

Let(X,τ) and (Y,τ') be topological spaces. A function $f : X \to Y$ is said to be:

- α -faintly continuous function in $x \in X$ if for each α-open V with $f(x) \in V$ there exists an open set U of X such that $x \in U$ and $f(U) \subseteq V$.

- α-faintly open function if for each open U of X, $f(U)$ is α-open in Y.
- α-continuous function in $x \in X$ if for each α-open V with $f(x) \in V$ there exists an α-open set U of X such that $x \in U$ and $f(U) \subseteq V$.
- α-open function if for each α-open U of X, $f(U)$ is α-open in V.

We denote $C^\alpha(X,Y)$ the set of α-faintly continuous of X on Y. Notice that $C^\alpha(X,Y) = C((X,\tau),(Y,\tau'^\alpha)$ Let A be a topological space. The transpose map $\hat{T} : A \to C^\alpha(X,Y)$ of a function $T : A \times X \to Y$, is given by $\hat{T}(a) = T_a$, being $T_a(x) = T(a,x)$. By δ_{XY} we mean the evaluation map: $\delta_{XY} : C^\alpha(X,Y) \times X \to Y$, $\delta_{XY}(f,x) = f(x)$.

Definition 4.1. A topology on $C^\alpha(X,Y)$ is called:

- α-strong if $T : A \times X \to Y$ α-faintly open $\Longrightarrow \hat{T} : A \to C^\alpha(X,Y)$ α-faintly open.
- α-weak if $\hat{T} : A \to C^\alpha(X,Y)$ α-faintly open $\Longrightarrow T : A \times X \to Y$ α-faintly open.
- α-exponential if it is α-weak and α-strong.

Proposition 4.2. *If a topology on $C^\alpha(X,Y)$ is α-weak, then the evaluation map δ_{XY} is α-faintly open.*

Proof. Take $A = C^\alpha(X,Y)$ and $T = \delta_{XY}$. So $\hat{T}(f) = (\hat{\delta}_{XY})(f) = f$ and then $\hat{T}$ is the identity map on $C^\alpha(X,Y)$. Since an open set is α-open set, it is followed that $\hat{T}$ is α-faintly open. Now, applying the hypothesis, it is obtained that $T = \delta_{XY}$ is α-faintly open. □

Proposition 4.3. *If the evaluation map δ_{XY} is an α-open function, then the topology on $C^\alpha(X,Y)$ is α-weak.*

Proof. Assume that $\hat{T} : A \to C^\alpha(X,Y)$, α-faintly open, where $\hat{T}$ is the transpose map of T. Take an open set U_A in A and an open set U_X in X. By hypothesis $\hat{T}(U_A)$ is α-open in $C^\alpha(X,Y)$. Then $\hat{T}(U_A) \times U_X$ is α-open, and thus $\delta_{XY}(\hat{T}(U_A) \times U_X)$ is α-open in Y. Consider $(a,x) \in U_A \times U_X$, $\delta_{XY}(\hat{T}(a),x) = \hat{T}(a)(x) = T(a,x)$. So $T(U_A \times U_X) = \delta_{XY}(\hat{T}(U_A) \times U_X)$ is α-open in Y. □

Lemma 4.4. *The following properties are fulfilled:*

(1) *Let τ_W and τ_S be an α-weak and α-strong topology on $C^\alpha(X,Y)$, respectively. Then $\tau_W \subseteq (\tau_S)^\alpha$.*
(2) *If τ is a topology on $C^\alpha(X,Y)$ such that $(\tau_S)^\alpha \subseteq \tau^\alpha$ where τ_S is an α-strong topology then τ is α-strong topology.*
(3) *If τ is a topology on $C^\alpha(X,Y)$ such that $\tau^\alpha \subseteq (\tau_w)^\alpha$ where τ_w is an α-weak topology then τ is an α-weak topology.*

Proof. 1.Applying proposition 4.2, it is obtained that the evaluation map $\delta_{XY} : (C^\alpha(X,Y),\tau_W) \times X \to Y$ is α-faintly open. From definition of τ_S, $\hat{\delta_{XY}} : A \to (C^\alpha(X,Y),\tau_S)$, where $A = (C^\alpha(X,Y),\tau_W)$ is α-faintly open. As $\hat{\delta_{XY}}$ is the identity map then $\tau_W \subseteq (\tau_S)^\alpha$.

2.Let τ be a topology on $C^\alpha(X,Y)$ such that $(\tau_S)^\alpha \subseteq \tau^\alpha$ and assume that $T : A \times X \to Y$ is α-faintly open then $\hat{T} : A \to (C^\alpha(X,Y),\tau_S)$, α-faintly open. As $(\tau_S)^\alpha \subseteq \tau^\alpha$ then $id : (C^\alpha(X,Y), \tau_S) \to (C^\alpha(X,Y), \tau)$ is α-open. So, $\hat{T} : A \to (C^\alpha(X,Y), \tau)$, is α-faintly open.

3. Let τ be a topology on $C^\alpha(X,Y)$ such that $\tau^\alpha \subseteq (\tau_W)^\alpha$ and assume that $\hat{T} : A \to (C^\alpha(X,Y), \tau)$, is α-faintly open. As $\tau^\alpha \subseteq (\tau_W)^\alpha$ then $id : (C^\alpha(X,Y), \tau) \to (C^\alpha(X,Y), \tau_W)$ is α-open then $\hat{T} : A \to (C^\alpha(X,Y), \tau_W)$ is α-faintly open. By definition, $T : A \times X \to Y$, α-faintly open. Therefore, τ is α-weak. □

The α-punctual convergence topology on $C^\alpha(X,Y)$ $\tau_{\alpha p}$ stands for the topology generated by the sets $W(x,V) = \{f : f(x) \in V\}$, where $x \in X$ and V is an α-open in Y. The α-open compact topology on $C^\alpha(X,Y)$, $\tau_{\alpha co}$ stands for the topology generated by the sets $W(K,V) = \{f : f(K) \subseteq V\}$, where $K \subseteq X$ is compact and V is an α-open in Y.

Notice that each $W(x,V)$ is α-open since it is open. Additionally, if $f \in W(K,V)$ then $f(K)$ is α-compact.

Proposition 4.5. $\tau_{\alpha p} \subseteq \tau_{\alpha co}$

Proof. Each $W(x,V) \in \tau_{\alpha co}$, since $\{x\}$ is a compact set of X for all $x \in X$. □

Example 4.6. Let X be the Niemytzki plane (or Moore plane), that is $X = \mathbb{R} \times [0,\infty)$ and a base for a topology contains two types of sets: open disk centered at (a,b) with $b > 0$ and $\{(a,0)\} \cup \{(x,y) : (x-a)^2 + (y-r)^2 < r^2\}, r > 0$ for $b = 0$.

Take $Y = \mathbb{R}$ with usual topology. We define the convergence topology on α-open with the base:

$\mathcal{B} = \{W(N,U) \subseteq C^\theta(X,Y) : N \subseteq X, N \text{ is } \alpha\!-\!open, U \subseteq Y, Y \text{ is } open\}$, where

$$W(N,U) = \{f \in C^\alpha(X,Y) : f(N) \subseteq U\}.$$

The evaluation map is α-open with this topology on $C^\alpha(X,Y)$.

5. Composition operators on spaces of α-faintly continuous functions

Let $\varphi : X \to X$ be a continuous function, the composition operator $C_\varphi : C^\alpha(X,Y) \to C^\alpha(X,Y)$ is defined by: $C_\varphi(f) = f \circ \varphi$

Theorem 5.1. *Consider $C^\alpha(X,Y)$ endowed of an α-exponential topology τ. Suppose φ homeomorphism. Then $C_\varphi : C^\alpha(X,Y) \to C^\alpha(X,Y)$ is α-faintly open.*

Proof. C_φ is well defined. Fix $f_0 \in C^\alpha(X,Y)$ and $x_0 \in X$. Let U_{f_0} open with $f_0 \in U_{f_0}$ and U_{x_0} open in X with $x_0 \in U_{x_0}$. Then $\varphi(U_{x_0})$ is open in X. As τ is α-weak then $\delta_{XY} : C^\alpha(X,Y) \times X \to Y$ is α-faintly open in $(f_0, \varphi(x_0))$. So, $U_{f_0} \times \varphi(U_{x_0})$ is open in $C^\alpha(X,Y) \times X$, and then $\delta_{XY}(U_{f_0} \times \varphi(U_{x_0}))$ is α-open in Y. Let $T : C^\alpha(X,Y) \times X \to Y$ defined by $T(f,x) = C_\varphi(f)(x) = \delta_{XY}(C_\varphi(f), x)$. Hence, $T(U_{f_0} \times U_{x_0}) = \delta_{XY}(U_{f_0}) \times \varphi(U_{x_0})$. So, T is α-faintly open. As τ is α-strong then $\hat{T} : C^\alpha(X,Y) \to C^\alpha(X,Y)$ is α-faintly open, $\hat{T}(f) = C_\varphi(f)$. □

Remark 5.2. If the space $C^\theta(X,Y)$ is endowed with a θ-exponential topology then the composition operator C_φ on $C^\theta(X,Y)$ is θ-faintly continuous, whereas if the space $C^\alpha(X,Y)$ is endowed with an α-exponential topology then the composition operator C_φ on $C^\alpha(X,Y)$ is α-faintly open. The properties of the transposed map associated with the evaluation map are key in the proof. The evaluation map is θ-faintly continuous if the topology on $C^\theta(X,Y)$ is θ-strong and is α-faintly open if $C^\alpha(X,Y)$ is α-weak. As expected, the θ punctual convergence topology on $C^\theta(X,Y)$ is θ-weak and the α-punctual convergence topology on $C^\alpha(X,Y)$ is contained in the α-open compact topology.

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