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Article

Extension of Meir-Keeler-Khan $(\psi - \alpha)$ Type Contraction in Partial Metric Space

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Abstract: In numerous scientific and engineering domains, fractional-order derivatives and integral operators are frequently used to represent many complex phenomena. They also have numerous practical applications in the area of fixed point iteration. In this article, we introduce the notion of generalized Meir-Keeler-Khan-Rational type $(\psi - \alpha)$-contraction mapping and propose fixed point results in partial metric spaces. Our proposed results extend, unify, and generalize existing findings in the literature. In regards to applicability, we provide evidence for the existence of a solution for the fractional-order differential operator. In addition, the solution of the integral equation and its uniqueness are also discussed. Finally, we conclude that our results are superior and generalized as compared to the existing ones.

Keywords: metric space; fixed point; fractional differential operator; non-linear equation; order of convergence

MSC: 47H10; 54H25

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1. Introduction

Many generalizations of the usual metric space have been introduced by many authors in the literature, and PMS (partial metric space) is one of them. Matthews [1] defined PMS in 1992 by considering the fact that self-distance is not necessary to be zero. He proposed the notion of PMS in his research on the denotational semantics of dataflow networks by proving that the BCP (Banach contraction principle) can be extended to the PMS for use in the verification of programs. The main purpose of deriving PMS is to transfer mathematical techniques into computer science. Inspired by this innovative idea of PMS, various authors have worked on this space and its physical properties, and also proved many FPRs (fixed point results) for single and multivalued-maps. Fixed point theory is simply based on the solution of a simple equation $\hbar\nu = \nu$ for a SM (self-map) $\hbar$ defined on a non-empty set χ. It is the most effective and successful technique for solving many mathematical problems, such as differential and integral equations which appear in economics, physics, chemistry, game theory, etc. The FP (fixed point) problem was first seen in the solution of a differential equation with an initial value. Liouville [2] was the first to obtain the solution of the FP equation by solving a differential equation with an initial value in 1837. Later, Picard [3] proposed the aforementioned method systematically and simplified the differential equation. Further, Banach [4] derived a celebrated FP theorem in complete metric space by developing the successive approximation method. This celebrated FP result was characterized by Caccioppoli [5], who showed that in a complete metric space, there is a UFP (unique fixed point) for each contraction. This result is also known as the

Picard–Banach FP theorem, the Banach–Caccioppoli FP theorem, Banach's FP theorem, and BCP.

After that, as a result of extending and generalizing the BCP, various FPRs have been proved by many authors in numerous spaces. Jaggi [6] was the first to establish a FP result by considering the rational expression in 1977. In 2018, Karapinar generated the idea of interpolative type contraction, and many more results have been proven in the context of interpolative type contraction by linking it with interpolative theory [7–10]. Inspired by Mitrovic et al. [11], Karapinar [8] introduced the notion of hybrid contraction by combining the ideas of interpolative type contractions and Reich type contractions in 2019. In 2021, Reena Jain et al. [12] defined an implicit contractive condition by an implicit relation on rational quasi PMS and derived periodic, and FP result. They also attained sufficient conditions for existing the unique positive solution of a non-linear matrix equation. Using auxiliary functions, the existence of FP with its uniqueness was proven by Kumar et al. [13] in 2021. Later in 2022, Nuseir et al. established a FP result for SM with some contractive conditions in partially ordered "E" MS. Saluja [14] attained a few common FPRs using auxiliary functions in a complete weak PMS.

In 1969, Meir and Keeler [15] demonstrated the concept of Meir-Keeler type contraction. This contraction is generalized by many authors in numerous type of spaces [16–19]. Motivated by these studies, Aydi et al. [9] defined GMKC (generalized Meir-Keeler type contraction) on PMS and they demonstrated that in 0-complete MS, a UFP exists for an orbitally continuous SM that satisfies the requirements of a GMKC. Later, Redjel et al. [20] derived the idea of $(\psi - \alpha)$ Meir-Keeler-Khan mapping in metric space. Further, in 2018, Kumar and Araci [21] established the notion of GMKK$(\psi - \alpha)$C (generalized Meir-Keeler-Khan type $(\psi - \alpha)$ contraction), which includes the α-admissibility of the function. Here, a FP result is established in complete PMS for GMKK$(\psi - \alpha)$C using the continuity of α-function, that is:

"In a complete PMS (χ, p), GMKK$(\psi - \alpha)$C has a FP, if

1. there exists $\nu_0 \in \chi$ in such a manner that $\alpha(\nu_0, \nu_0) \geq 1$,
2. if $\alpha(\nu_k, \omega_k) \geq 1$ for each $k \in N$, then $\lim_{k\to\infty} \alpha(\nu_k, \omega_k) \geq 1$,
3. $\alpha : \chi^2 \to R^+$ is a continuous function in each coordinate".

Moreover, in 2019, Karapinar and Fulga [8] initiated the notion of hybrid type contraction in complete metric space by combining the idea of Jaggi type contraction with interpolative type contraction and stated FPRs using the continuity of the SM. Their proposed results state that "A Jaggi type hybrid contraction $\hbar : \chi \to \chi$ possesses a FP in a complete metric space (χ, d), if $\hbar$ is continuous and attained a UFP if $\hbar^p$ is continuous for some integer $p > 1$". Furthermore, they established the solution of FDE(fractional differential equation) in the framework of their demonstrated result.

Inspired by [8,21], we have related the idea of Jaggi type hybrid contraction with GMKK$(\psi - \alpha)$C and stated GMKKR(ψ)C (generalized Meir-Keeler-Khan-Rational type ψ-contraction) and GMKKR$(\psi - \alpha)$C (generalized Meir-Keeler-Khan-Rational type $(\psi - \alpha)$ contraction) and proved FPRs in complete PMS by relaxing the condition of continuity of the α-function and the continuity of SM. Additionally, we have provided illustrative examples in support of our result. Furthermore, we have applied our proposed result in integral and fractional calculus to obtained the existence and uniqueness of solutions of VIE (Volterra integral equation), Caputo type FDE, and Riemann-Liouville type FDO (fractional differential operator), and to justify all of these results, we have proposed examples for each result.

Beginning with the introduction in Section 1, we have given some fundamental definitions in Section 2. In Section 3, we have proposed the concept of GMKKR(ψ)C, GMKKR$(\psi - \alpha)$C and proved FPRs with their uniqueness in complete PMS. To justify our result attained in Section 3, we have given examples. We have also worked on a VIE of second kind, a FDE of Caputo type, and a FDO of Riemann-Liouville type and attained the existence and uniqueness of solutions for all of these in the framework of our main theorem, with examples in Section 4.

2. Definitions

Here, a few fundamental definitions are presented pertaining to our work.

Definition 1. *Ψ is the family of ψ functions. $\psi : [0,\infty) \to [0,\infty)$ is a function satisfying the below prepositions:*

1. *ψ is non-decreasing and continuous,*
2. *$\Sigma_{n=1}^{\infty}\psi^n(t) < \infty$ for each $t > 0$,*
3. *$\psi(t) < t$ for all $t > 0$ and $\psi(t) = 0$ if and only if $t = 0$.*

Definition 2. *Consider a non-empty set χ. Define a SM $T : \chi \to \chi$ and a function $\alpha : X \times X \to [0,\infty)$. Then, T is said to be $\alpha-$admissible, if*

$$\forall x, y \in X, \alpha(x,y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 3 ([8])**.** *Let $\hbar$ be a SM defined on a MS (χ, d). Then, $\hbar$ is known as Jaggi type hybrid contraction if there exists $\psi \in \Psi$ (defined in Definition 1) in such a way that*

$$d(\hbar\nu, \hbar\omega) \leq \psi(\zeta^s_\hbar(\nu,\omega)), \tag{1}$$

for every distinct $\nu, \omega \in \chi, s \geq 0$. Now, the function $\zeta^s_\hbar(\nu,\omega)$ is defined as

$$\zeta^s_\hbar(\nu,\omega) = \begin{cases} \left(\sigma_1\left(\frac{p(\nu,\hbar\nu)p(\omega,\hbar\omega)}{p(\nu,\omega)}\right)^s + \sigma_2(p(\nu,\omega))^s\right)^{\frac{1}{s}}, \ s > 0,\ \nu,\omega \in \chi(\nu \neq \omega), \\ (p(\nu,\hbar\nu))^{\sigma_1}(p(\omega,\hbar\omega))^{\sigma_2},\ s = 0,\ \nu,\omega \in \chi/F_\hbar(\chi), \end{cases}$$

with $F_\hbar(\chi) = \{z \in \chi : \hbar z = z\}$ and $\sigma_1, \sigma_2 \geq 0$ such that $\sigma_1 + \sigma_2 = 1$.

Definition 4 ([21])**.** *Define a SM $\hbar$ on a PMC (χ, p). Then, $\hbar$ is called GMKK$(\psi-\alpha)$C, if*

1. *$\hbar$ is α-admissible,*
2. *for each $\epsilon > 0$, there exists some $\delta > 0$ such that*

$$\epsilon \leq \psi\left(\frac{p(\nu,\hbar\nu)p(\nu,\hbar\omega) + p(\omega,\hbar\omega)p(\omega,\hbar\nu)}{p(\nu,\hbar\omega) + p(\omega,\hbar\nu)}\right) < \epsilon + \delta \Rightarrow \alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega) < \epsilon, \tag{2}$$

for each $\nu, \omega \in \chi$. The ψ-function and α-admissibility of the function is defined in Definitions 1 and 2 respectively.

Remark 1 ([21])**.** *If $\hbar$ is GMKK$(\psi-\alpha)$C, then*

$$\alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega) \leq \psi\left(\frac{p(\nu,\hbar\nu)p(\nu,\hbar\omega) + p(\omega,\hbar\omega)p(\omega,\hbar\nu)}{p(\nu,\hbar\omega) + p(\omega,\hbar\nu)}\right), \forall \nu,\omega \in \chi. \tag{3}$$

3. Development of Extension of Meir-Keeler-Khan $(\psi-\alpha)$ Type Contraction and Related FPRs

Here, we have proposed GMKKRψC and GMKKR$(\psi-\alpha)$C in PMS as an extension of Meir-Keeler-Khan $(\psi-\alpha)$ type contraction. Additionally, we have established FPRs for these proposed contractions.

Definition 5. *Assume $\hbar$ be a SM defined on a PMS (χ, p). Then, $\hbar$ is called GMKKR(ψ)C, if for each $\epsilon > 0$, there exists some $\delta > 0$ satisfying*

$$\epsilon \leq \psi(\zeta^s_\hbar(\nu,\omega)) < \epsilon + \delta \Rightarrow p(\hbar\nu,\hbar\omega) < \epsilon. \tag{4}$$

Here, the functions ψ and $\zeta^s_\hbar(\nu,\omega)$ are the same as defined in Definitions 1 and 3, respectively.

Definition 6. *Suppose $\hbar$ is an SM defined on a PMS (χ, p). Then, $\hbar$ is called GMKKR$(\psi - \alpha)$C, if*
1. *$\hbar$ is α-admissible,*
2. *for each $\epsilon > 0$, there exists some $\delta > 0$ in such a manner that*

$$\epsilon \leq \psi(\zeta^s_{\hbar}(\nu, \omega)) < \epsilon + \delta \Rightarrow \alpha(\nu, \nu)\alpha(\omega, \omega)p(\hbar\nu, \hbar\omega) < \epsilon, \tag{5}$$

The functions ψ and $\zeta^s_{\hbar}(\nu, \omega)$ are same as defined in Definitions 1 and 3 respectively, and α-admissibility of the function is defined in Definition 2.

Remark 2. *If $\hbar$ is GMKKR$(\psi - \alpha)$C, then*

$$\alpha(\nu, \nu)\alpha(\omega, \omega)p(\hbar\nu, \hbar\omega) \leq \psi(\zeta^s_{\hbar}(\nu, \omega)), \forall \nu, \omega \in \chi. \tag{6}$$

Theorem 1. *Suppose (χ, p) be a complete PMS and $\hbar : \chi \to \chi$ be a GMKKR$(\psi - \alpha)$C. Then, $\hbar$ has a UFP, if*
1. *there exists $\nu_0 \in \chi$ such that $\alpha(\nu_0, \nu_0) \geq 1$,*
2. *$\psi(t) \leq \frac{2t}{3}, \forall t$.*

Here, the function ψ is same as defined in Definition 1 and $\alpha : X \times X \to [0, \infty)$.

Proof. Construct a Picard sequence for $\nu_0 \in \chi$ as

$$\nu_{k+1} = \hbar\nu_k, \forall k = 0, 1, 2, \dots \tag{7}$$

From given condition 1, we have

$$\alpha(\nu_0, \nu_0) \geq 1. \tag{8}$$

Since $\hbar$ is GMKKR$(\psi - \alpha)$C, $\hbar$ is α-admissible. Therefore, using Definition 2, we attain

$$\alpha(\hbar\nu_0, \hbar\nu_0) \geq 1 \Rightarrow \alpha(\nu_1, \nu_1) \geq 1. \quad \text{[using (7)]}$$

By proceeding with the process in the same manner, we obtain

$$\alpha(\nu_k, \nu_k) \geq 1, \forall k = 0, 1, 2, \dots \tag{9}$$

Here, we have two cases:
Case (I): If $\nu_{k_0} = \nu_{k_0+1}$ for some $k_0 \in$ N , then, $\nu_{k_0} = \hbar\nu_{k_0}$. Therefore, ν_{k_0} is FP of $\hbar$.
Case (II): If $\nu_{k_0} \neq \nu_{k_0+1}$, then again we have two cases:
case (a): For $s > 0$:

From definition of ψ (defined in Definition 1), we have

$$\psi(t) > 0, \forall t > 0. \tag{10}$$

$$\text{Therefore, } \psi(\zeta^s_{\hbar}(\nu, \omega)) > 0,$$

$$\psi\left(\sigma_1\left(\frac{p(\nu, \hbar\nu)p(\omega, \hbar\omega)}{p(\nu, \omega)}\right)^s + \sigma_2(p(\nu, \omega))^s\right)^{\frac{1}{s}} > 0.$$

For $\nu = \nu_k$ and $\omega = \nu_{k+1}$, the above inequality becomes

$$\psi\left(\sigma_1\left(\frac{p(\nu_k, \hbar\nu_k)p(\nu_{k+1}, \hbar\nu_{k+1})}{p(\nu_k, \nu_{k+1})}\right)^s + \sigma_2(p(\nu_k, \nu_{k+1}))^s\right)^{\frac{1}{s}} > 0,$$

$$\psi\left(\sigma_1\left(\frac{p(\nu_k, \nu_{k+1})p(\nu_{k+1}, \nu_{k+2})}{p(\nu_k, \nu_{k+1})}\right)^s + \sigma_2(p(\nu_k, \nu_{k+1}))^s\right)^{\frac{1}{s}} > 0,$$

$$\psi(\sigma_1(p(\nu_{k+1}, \nu_{k+2}))^s + \sigma_2(p(\nu_k, \nu_{k+1}))^s)^{\frac{1}{s}} > 0.$$

Now, if $p(\nu_k, \nu_{k+1}) \leq p(\nu_{k+1}, \nu_{k+2})$. Then, using above inequality, we can say

$$\begin{aligned} \psi((\sigma_1 + \sigma_2)(p(\nu_{k+1}, \nu_{k+2}))^s)^{\frac{1}{s}} &> 0, \\ \psi(p(\nu_{k+1}, \nu_{k+2})) &> 0. \quad [\text{as } \sigma_1 + \sigma_2 = 1] \end{aligned}$$

From Remark 2 (for $\nu = \nu_k$ and $\omega = \nu_{k+1}$), we have

$$\alpha(\nu_k, \nu_k)\alpha(\nu_{k+1}, \nu_{k+1})p(\hbar\nu_k, \hbar\nu_{k+1}) \leq \psi(\zeta_\hbar^s(\nu_k, \nu_{k+1})). \tag{11}$$

Using Equations (7) and (9) in above expression (11), we obtain

$$\begin{aligned} p(\nu_{k+1}, \nu_{k+2}) &\leq \psi(\zeta_\hbar^s(\nu_k, \nu_{k+1})), \\ &\leq \psi(p(\nu_{k+1}, \nu_{k+2})), \\ &< p(\nu_{k+1}, \nu_{k+2}), \end{aligned}$$

which is a contradiction. Therefore, our assumption is wrong. Hence

$$p(\nu_{k+1}, \nu_{k+2}) \leq p(\nu_k, \nu_{k+1}). \tag{12}$$

Thus, sequence $\{\nu_k\}$ is a decreasing sequence and hence, converges to some $\epsilon \geq 0$, i.e.,

$$\lim_{k\to\infty} p(\nu_k, \nu_{k+1}) = \epsilon, \tag{13}$$

$$\text{where } \epsilon = \inf\{p(\nu_k, \nu_{k+1})\}, k \in \mathbb{N} \cup \{0\}.$$

Now, we will show that $\epsilon = 0$. On the contrary, suppose that $\epsilon \neq 0$. Then, using Definition 4 for $\mu = \nu_k$ and $\nu = \nu_{k+1}$, we get

$$\begin{aligned} p(\nu_{k+1}, \nu_{k+2}) &= p(\hbar\nu_k, \hbar\nu_{k+1}), \\ &\leq \alpha(\nu_k, \nu_k)\alpha(\nu_{k+1}, \nu_{k+1})p(\hbar\nu_k, \hbar\nu_{k+1}) < \epsilon, \quad [\text{using Equation (9)}] \end{aligned}$$

which is a contradiction because $\epsilon = \inf\{p(\nu_k, \nu_{k+1})\}, k \in \mathbb{N} \cup \{0\}$. Thus, our supposition is wrong and therefore, $\epsilon = 0$. Hence, Equation (13) becomes

$$\lim_{k\to\infty} p(\nu_k, \nu_{k+1}) = 0, \forall k \in \mathbb{N}. \tag{14}$$

From the definition of partial metric, we have

$$0 \leq \lim_{k\to\infty} p(\nu_k, \nu_k) \leq \lim_{k\to\infty} p(\nu_k, \nu_{k+1}).$$

Using Equation (14) in the above inequality, we attain

$$\lim_{k\to\infty} p(\nu_k, \nu_k) = 0. \tag{15}$$

Further, we will demonstrate that sequence $\{\nu_k\}$ is a Cauchy sequence in PMS (χ, p). To show this, it is sufficient to prove that sequence $\{\nu_k\}$ is a Cauchy sequence in metric space (χ, d_p), where

$$d_p(\nu_k, \nu_{k+1}) = 2p(\nu_k, \nu_{k+1}) - p(\nu_k, \nu_k) - p(\nu_{k+1}, \nu_{k+1}). \tag{16}$$

Letting $k \to \infty$ and applying expressions (14) and (15) in Equation (16), we attain

$$\lim_{k\to\infty} d_p(\nu_k, \nu_{k+1}) = 0, \forall k \in \mathbb{N}. \tag{17}$$

Now, suppose that sequence $\{\nu_k\}$ is not a Cauchy sequence in metric space (χ, d_p). So, there exists a number $\eta > 0$ such that for any $c \in \mathbb{N}$, there are two numbers $n_c, m_c(n_c \geq m_c \geq c)$ satisfying

$$d_p(\nu_{m_c}, \nu_{n_c}) \geq \eta. \tag{18}$$

Additionally, for $m_c \geq c$, we can choose a small positive integer n_c in such a manner that for $n_c \geq m_c \geq c$, we have

$$d_p(\nu_{m_c}, \nu_{n_{c-2}}) < \eta. \tag{19}$$

From Equation (18), we have

$$\begin{aligned} \eta &\leq d_p(\nu_{m_c}, \nu_{n_c}), \\ &\leq d_p(\nu_{m_c}, \nu_{n_{c-2}}) + d_p(\nu_{n_{c-2}}, \nu_{n_{c-1}}) + d_p(\nu_{n_{c-1}}, \nu_{n_c}). \end{aligned}$$

Letting $c \to \infty$ and using Equations (17) and (19) in above inequality, we get

$$\lim_{c\to\infty} d_p(\nu_{m_c}, \nu_{n_c}) = \eta, \tag{20}$$

$$\begin{aligned} \text{Also, } \eta &\leq d_p(\nu_{m_c}, \nu_{n_c}), \\ &\leq d_p(\nu_{m_c}, \nu_{m_{c+1}}) + d_p(\nu_{m_{c+1}}, \nu_{n_{c+1}}) + d_p(\nu_{n_{c+1}}, \nu_{n_c}), \\ &\leq d_p(\nu_{m_c}, \nu_{m_{c+1}}) + d_p(\nu_{m_{c+1}}, \nu_{m_c}) + d_p(\nu_{m_c}, \nu_{n_c}) + d_p(\nu_{n_c}, \nu_{n_{c+1}}) + d_p(\nu_{n_{c+1}}, \nu_{n_c}). \end{aligned}$$

Letting $c \to \infty$ and substituting Equations (17) and (20) in above inequality, we obtain

$$\lim_{c\to\infty} d_p(\nu_{m_{c+1}}, \nu_{n_{c+1}}) = \eta. \tag{21}$$

Further, for $\nu = \nu_{m_c}$ and $\nu = \nu_{n_c}$, Equation (16) becomes

$$\begin{aligned} d_p(\nu_{m_c}, \nu_{n_c}) &= 2p(\nu_{m_c}, \nu_{n_c}) - p(\nu_{m_c}, \nu_{m_c}) - p(\nu_{n_c}, \nu_{n_c}), \\ d_p(\nu_{m_{c+1}}, \nu_{n_{c+1}}) &= 2p(\nu_{m_{c+1}}, \nu_{n_{c+1}}) - p(\nu_{m_{c+1}}, \nu_{m_{c+1}}) - p(\nu_{n_{c+1}}, \nu_{n_{c+1}}). \end{aligned}$$

Letting $c \to \infty$ and applying Equations (17), (20) and (21) in above expression, we get

$$\lim_{c\to\infty} p(\nu_{m_c}, \nu nc) = \lim_{c\to\infty} p(\nu_{m_{c+1}}, \nu_{n_{c+1}}) = \frac{\eta}{2}, \tag{22}$$

$$\begin{aligned} p(\nu_{m_{c+1}}, \nu_{n_{c+1}}) &= p(\hbar\nu_{m_c}, \hbar\nu_{m_c}), \\ &\leq \alpha(\nu_{m_c}, \nu_{m_c})\alpha(\nu_{n_c}, \nu_{n_c})p(\hbar\nu_{m_c}, \hbar\nu_{m_c}), \\ &\leq \psi(\zeta^s_{\hbar}(\nu_{m_c}, \nu_{n_c})), \quad \text{[using Equation (6)]} \\ &= \psi\left(\sigma_1\left(\frac{p(\nu_{m_c}, \hbar\nu_{m_c})p(\nu_{n_c}, \hbar\nu_{n_c})}{p(\nu_{m_c}, \nu_{n_c})}\right)^s + \sigma_2(p(\nu_{m_c}, \nu_{n_c}))^s\right)^{\frac{1}{s}}, \\ &\leq \psi\left(\sigma_1\left(\frac{p(\nu_{m_c}, \nu_{m_{c+1}})p(\nu_{n_c}, \nu_{n_{c+1}})}{p(\nu_{m_c}, \nu_{n_c})}\right)^s + \sigma_2(p(\nu_{m_c}, \nu_{n_c}))^s\right)^{\frac{1}{s}}. \end{aligned}$$

Letting $c \to \infty$ and substituting Equations (14) and (22) in the above inequality, we get

$$\begin{aligned} \frac{\eta}{2} &\leq \lim_{c\to\infty} \psi(\sigma_2(p(\nu_{m_c}, \nu_{n_c}))^s)^{\frac{1}{s}}, \\ \frac{\eta}{2} &\leq \lim_{c\to\infty} \psi(p(\nu_{m_c}, \nu_{n_c})), \quad [\text{as } \sigma_2 \leq 1] \\ \frac{\eta}{2} &< \lim_{c\to\infty} p(\nu_{m_c}, \nu_{n_c}), \quad [\text{as } \psi(t) < t \text{ from Definition 1}] \\ \frac{\eta}{2} &< \frac{\eta}{2}, \end{aligned}$$

which is not true. Therefore, our supposition is wrong. Thus, sequence (ν_k) is a Cauchy sequence in metric space (χ, d_p) and hence, a Cauchy sequence in PMS (χ, p). Further, completeness of PMS (χ, p) implies completeness of metric space (χ, d_p). Therefore, Cauchy sequence $\{\nu_k\}$ converges in metric space (χ, d_p). Hence, there exists a number $z \in \chi$ in such a manner that $\lim_{k\to\infty} \nu_k = z$, that is

$$\lim_{k\to\infty} d_p(\nu_k, z) = \lim_{k,l\to\infty} d_p(\nu_k, \nu_l) = 0. \tag{23}$$

Now, we know that

$$\lim_{k\to\infty} d_p(z, v_k) = 0 \Leftrightarrow \lim_{k\to\infty} p(z, v_k) = \lim_{k,l\to\infty} p(v_k, v_l) = p(z, z).$$

$$\text{Thus, } p(z,z) = \lim_{k\to\infty} p(\nu_k, z) = \lim_{k,l\to\infty} p(\nu_k, \nu_l) = 0, \quad \text{[using Equation (15)]} \tag{24}$$

$$\begin{aligned}
p(\nu_{k+1}, \hbar z) &= p(\hbar\nu_k, \hbar z), \\
&\le \alpha(\nu_k, \nu_k)\alpha(z,z)p(\hbar\nu_k, \hbar z), \\
&\le \psi(\zeta_\hbar^s(\nu_k, z)), \quad \text{[using Equation (6)]} \\
&= \psi\left(\sigma_1\left(\frac{p(\nu_k, \hbar\nu_k)p(z, \hbar z)}{p(\nu_k, z)}\right)^s + \sigma_2(p(\nu_k, z))^s\right)^{\frac{1}{s}}, \\
&\le \psi\left(\sigma_1\left(\frac{(p(\nu_k, z) + p(z, \nu_{k+1}) - P(z,z))p(z, \hbar z)}{p(\nu_k, z)}\right)^s + \sigma_2(p(\nu_k, z))^s\right)^{\frac{1}{s}}.
\end{aligned}$$

Letting $k \to \infty$ and substituting Equation (24) in the above inequality, we get

$$\begin{aligned}
\lim_{k\to\infty} p(z, \hbar z) &\le \psi(\sigma_1(p(z, \hbar z))^s)^{\frac{1}{s}}, \\
&= \psi\left(\sigma_1^{\frac{1}{s}}(p(z, \hbar z))\right), \\
&\le \frac{2\sigma_1^{\frac{1}{s}} p(z, \hbar z)}{3}, \\
&\le \frac{2p(z, \hbar z)}{3}, \quad [\text{as } \sigma_1 \le 1] \\
3p(z, \hbar z) &\le 2p(z, \hbar z), \\
p(z, \hbar z) &= 0, \\
\hbar z &= z.
\end{aligned}$$

Therefore, z is FP of $\hbar$.

Uniqueness of FP:

Assume $z_1 (\neq z)$ be another FP of $\hbar$, then

$$\begin{aligned}
\hbar z_1 &= z_1, \\
p(z, z_1) &= p(\hbar z, \hbar z_1), \\
&\le \alpha(z,z)\alpha(z_1, z_1)p(\hbar z, \hbar z_1), \\
&\le \psi(\zeta_\hbar^s(z, z_1)), \quad \text{[using Equation (6)]}
\end{aligned}$$

$$\begin{aligned}
p(z,z_1) &= \psi\left(\sigma_1\left(\frac{p(z,\hbar z)p(z_1,\hbar z_1)}{p(z,z_1)}\right)^s + \sigma_2(p(z,z_1))^s\right)^{\frac{1}{s}},\\
&= \psi\left(\sigma_1\left(\frac{p(z,z)p(z_1,z_1)}{p(z,z_1)}\right)^s + \sigma_2(p(z,z_1))^s\right)^{\frac{1}{s}},\\
&= \psi(\sigma_2(p(z,z_1))^s)^{\frac{1}{s}}, \quad [\text{as } p(z,z)=0 \text{ from Equation (24)}]\\
&= \psi(\sigma_2^{\frac{1}{s}} p(z,z_1)),\\
&\leq \frac{2\sigma_2^{\frac{1}{s}} p(z,z_1)}{3},\\
&\leq \frac{2p(z,z_1)}{3}, \quad [\text{as } \sigma_2 \leq 1]\\
3p(z,z_1) &\leq 2p(z,z_1),\\
p(z,z_1) &\leq 0,\\
p(z,z_1) &= 0. \qquad (25)\\
p(z_1,z_1) &= p(\hbar z_1,\hbar z_1),\\
&\leq \alpha(z_1,z_1)\alpha(z_1,z_1)p(\hbar z_1,\hbar z_1),\\
&\leq \psi(\zeta_\hbar^s(z_1,z_1)), \quad [\text{using Equation (6)}]\\
&= \psi\left(\sigma_1\left(\frac{p(z_1,\hbar z_1)p(z_1,\hbar z_1)}{p(z_1,z_1)}\right)^s + \sigma_2(p(z_1,z_1))^s\right)^{\frac{1}{s}},\\
&= \psi\left(\sigma_1\left(\frac{p(z_1,z_1)p(z_1,z_1)}{p(z_1,z_1)}\right)^s + \sigma_2(p(z_1,z_1))^s\right)^{\frac{1}{s}},\\
&= \psi((\sigma_1+\sigma_2)(p(\sigma_1,z_1))^s)^{\frac{1}{s}},\\
&\leq \frac{2p(z,z_1)}{3}, \quad [\text{as } \sigma_1+\sigma_2 = 1]\\
3p(z_1,z_1) &\leq 2p(z_1,z_1),\\
p(z_1,z_1) &= 0. \qquad (26)
\end{aligned}$$

From Equations (24)–(26), we obtain

$$p(z,z) = p(z,z_1) = p(z_1,z_1) = 0.$$

Therefore, from the definition of the partial metric, we conclude that $z = z_1$. Hence, z is the UFP of $\hbar$.

case (b): For $s = 0$:

Since $\hbar$ is GMKKR$(\psi - \alpha)$C. So, for $\nu = \nu_k$ and $\omega = \nu_{k+1}$, we have

$$\begin{aligned}
p(\nu_{k+1},\nu_{k+2}) &= p(\hbar\nu_k,\hbar\nu_{k+1}),\\
&\leq \alpha(\nu_k,\nu_k)\alpha(\nu_{k+1},\nu_{k+1})p(\hbar\nu_k,\hbar\nu_{k+1}),\\
&\leq \psi(\zeta_\hbar^s(\nu_k,\nu_{k+1})), \quad [\text{ using Equation (6)}]\\
&= \psi((p(\nu_k,\nu_{k+1}))^{\sigma_1}(p(\nu_{k+1},\nu_{k+2}))^{\sigma_2}), \quad [\text{from Definition 3}]\\
&< (p(\nu_k,\nu_{k+1}))^{\sigma_1}(p(\nu_{k+1},\nu_{k+2}))^{\sigma_2}, \quad [\text{from Definition 1}]\\
(p(\nu_{k+1},\nu_{k+2}))^{1-\sigma_2} &< (p(\nu_k,\nu_{k+1}))^{\sigma_1},\\
(p(\nu_{k+1},\nu_{k+2}))^{\sigma_1} &< (p(\nu_k,\nu_{k+1}))^{\sigma_1},\\
p(\nu_{k+1},\nu_{k+2}) &< p(\nu_k,\nu_{k+1}).
\end{aligned}$$

Therefore, $\{\nu_k\}$ is a decreasing sequence and hence convergent to some $\epsilon \geq 0$. Further, applying the same procedure as in case (a), we get

$$\lim_{k\to\infty} p(\nu_k,\nu_{k+1}) = 0. \tag{27}$$

$$\begin{aligned} &\lim_{k\to\infty} d_p(\nu_k,\nu_k) = 0, \\ &\lim_{c\to\infty} d_p(\nu_{m_c},\nu_{n_c}) = \lim_{c\to\infty} d_p(\nu_{m_{c+1}},\nu_{n_{c+1}}) = \eta, \\ &\lim_{c\to\infty} p(\nu_{m_c},\nu_{n_c}) = \lim_{c\to\infty} p(\nu_{m_{c+1}},\nu_{n_{c+1}}) = \frac{\eta}{2}, \end{aligned} \tag{28}$$

$$\begin{aligned} p(\nu_{m_{c+1}},\nu_{n_{c+1}}) &= p(\hbar\nu_{m_c},\hbar\nu_{n_c}), \\ &\leq \alpha(\nu_{m_c},\nu_{m_c})\alpha(\nu_{n_c},\nu_{n_c})p(\hbar\nu_{m_c},\hbar\nu_{n_c}), \\ &\leq \psi(\zeta^s_\hbar(\nu_{m_c},\nu_{n_c})), \\ &= \psi((p(\nu_{m_c},\nu_{m_{c+1}}))^{\sigma_1}(p(\nu_{n_c},\nu_{n_{c+1}}))^{\sigma_2}). \end{aligned}$$

Letting $c \to \infty$ and using Equations (27) and (28) in above inequality, we obtain

$$\frac{\eta}{2} \leq \psi(0) = 0,$$

which is not true as $\eta > 0$. Again applying the sane procedure as in case (a), we get

$$p(z,z) = \lim_{k\to\infty} p(\nu_k,z) = \lim_{k,l\to\infty} p(\nu_k,\nu_l) = 0. \tag{29}$$

$$\begin{aligned} p(\nu_{k+1},\hbar z) &= p(\hbar\nu_k,\hbar z), \\ &\leq \alpha(\nu_k,\nu_k)\alpha(z,z)p(\hbar\nu_k,\hbar z), \\ &\leq \psi(\zeta^s_\hbar(\nu_k,z)), \quad \text{[using Equation (6)]} \\ &= \psi((p(\nu_k,\nu_{k+1}))^{\sigma_1}(p(z,\hbar z))^{\sigma_2}). \quad \text{[from definition of } \psi \text{ defined in Definition 1]} \end{aligned}$$

Letting $k \to \infty$ and using Equation (29), we attain

$$\begin{aligned} p(z,\hbar z) &= \psi(0) = 0, \\ \hbar z &= z. \end{aligned}$$

Hence, z is FP of $\hbar$.

Uniqueness of FP for case (b):

We can attain the uniqueness in the same manner as in case (a). □

Theorem 2. *Consider a complete PMS (χ,p) and a SM $\hbar:\chi\to\chi$. If $\hbar$ is GMKKR(ψ)C, then $\hbar$ has a UFP for $\psi(t) \leq \frac{2t}{3}, \forall t$.*

Proof. The result can be directly attained from above Theorem 1 by taking $\alpha(\nu,\omega) = 1$ for each $\nu,\omega \in \chi$. □

4. Numerical Results

Here, in this part, some examples are illustrated in support of our proposed Theorem 1 in Section 3.

Example 1. *Consider a complete PMS (χ,p) with $\chi = [0,1]$ and $p(\nu,\omega) = \max\{\nu,\omega\}$. Define a SM $\hbar:\chi\to\chi$ as*

$$\hbar\nu = \frac{\nu}{8},$$

and a function α as

$$\alpha(\nu,\omega) = \begin{cases} 1+\frac{\nu}{3}+\frac{\omega}{3}, \nu,\omega \neq 1, \\ 1, \nu=\omega=1, \end{cases}$$

Then, $\hbar$ has UFP for $s=2, \psi(t)=\frac{t}{2}$ and $\sigma_1=\sigma_2=\frac{1}{2}$.

Proof. To show that $\hbar$ has a UFP, it is sufficient to show that all the assumptions of our proposed Theorem 1 are satisfied.

1. Obviously, $\forall \nu,\omega \in \chi, \alpha(\nu,\omega) \geq 1$. Therefore, there exists $\nu_0 \in \chi$ in such a way that $\alpha(\nu_0,\nu_0) \geq 1$.
2. $\alpha(\nu,\omega) \geq 1 \Rightarrow \alpha(\hbar\nu,\hbar\omega) \geq 1, \forall \nu,\omega \in \chi$, which shows that $\hbar$ is $\alpha-$admissible.
3. Without loss of generality, suppose $\nu \geq \omega$. Therefore,

$$\begin{aligned} p(\nu,\omega) &= \max\{\nu,\omega\} = \omega, \\ p(\nu,\hbar\nu) &= \max\{\nu,\hbar\nu\} = \max\left\{\nu,\frac{\nu}{8}\right\} = \nu, \\ p(\omega,\hbar\omega) &= \max\{\omega,\hbar\omega\} = \max\left\{\omega,\frac{\omega}{8}\right\} = \omega, \\ p(\hbar\nu,\hbar\omega) &= \max\{\hbar\nu,\hbar\omega\} = \max\left\{\frac{\nu}{8},\frac{\omega}{8}\right\} = \frac{\nu}{8}, \\ \psi(\zeta^s_\hbar(\nu,\omega)) &= \psi\left(\sigma_1\left(\frac{p(\nu,\hbar\nu)p(\omega,\hbar\omega)}{p(\nu,\omega)}\right)^s + \sigma_2(p(\nu,\omega))^s\right)^{\frac{1}{s}}, \\ \psi(\zeta^s_\hbar(\nu,\omega)) &= \frac{(\nu^2+\omega^2)^{\frac{1}{2}}}{2\sqrt{2}} \geq 0, \forall \nu,\omega \in \chi, \end{aligned} \tag{30}$$

$$\alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega) = \begin{cases} \left(1+\frac{2\nu}{3}\right)\left(1+\frac{2\omega}{3}\right)\left(\frac{\nu}{8}\right), \nu,\omega \neq 1, \\ \left(\frac{\nu}{8}\right), \nu=\omega=1, \end{cases}$$

$$\begin{aligned} \alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega) &= (1+\nu)(1+\omega)\left(\frac{\nu}{8}\right), \\ &= \left(\frac{\nu(1+\nu+\omega+\nu\omega)}{8}\right). \end{aligned} \tag{31}$$

From Equations (30) and (31), we can clearly see that

$$\alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega) \leq \psi(\zeta^s_\hbar(\nu,\omega)), \forall \nu,\omega \in \chi,$$

which shows that $\hbar$ is GMKKR$(\psi-\alpha)$C.

Hence, every hypothesis of Theorem 1 is satisfied. Thus, $\hbar$ has a UFP 0. Now, since α is not continuous at $\nu=1$, UFP cannot be determined for FPRs in [19,21]. □

Example 2. *Assume a PMS (χ,p) with $\chi=[0,1]$ and $p(\nu,\omega)=\max\{\nu,\omega\}$. Define a SM $\hbar:\chi\to\hbar$ and function α as*

$$\hbar\nu = \begin{cases} 0, \nu \in \left[0,\frac{1}{2}\right), \\ \frac{1}{6}, \nu \in \left[\frac{1}{2},1\right] \end{cases}$$

$$\alpha(\nu,\omega)=\begin{cases}1+\frac{\nu}{6}+\frac{\omega}{6},\nu,\omega\in\left[0,\frac{1}{2}\right),\\ 1+\frac{\nu}{6}-\frac{\omega}{6},\nu,\omega\in\left[\frac{1}{2},1\right]\end{cases}$$

$\forall\nu,\omega\in\chi$. *Then,* $\hbar$ *has UFP* 0 *for* $\psi(t)=\frac{2t}{3},\sigma_1=\frac{1}{4},\sigma_2=\frac{3}{4}$ *and* $s=2$.

Proof. For this, we will establish that every assumption of Theorem 1 is satisfied.

1. Clearly $\alpha(\nu,\omega)\geq 1,\forall\nu,\omega\in\chi$. Thus, there exists some $\nu_0\in\chi$ in such a manner that $\alpha(\nu_0,\nu_0)\geq 1$.
2. $\alpha(\nu,\omega)\geq 1\Rightarrow\alpha(\hbar\nu,\hbar\omega)\geq 1,\forall\nu,\omega\in\chi$. Therefore, $\hbar$ is α-admissible.
3. Without loss of generality, suppose $\nu\geq\omega$. Now, we have two cases:

Case (I): If $\nu,\omega\in\left[0,\frac{1}{2}\right)$, then

$$\begin{aligned}p(\nu,\omega)&=\max\{\nu,\omega\}=\nu,\\ p(\nu,\hbar\nu)&=\max\{\nu,\hbar\nu\}=\max\{\nu,0\}=\nu,\\ p(\omega,\hbar\omega)&=\max\{\omega,\hbar\omega\}=\max\{\omega,0\}=\omega,\\ p(\hbar\nu,\hbar v)&=\max\{\hbar\nu,\hbar\omega\}=\max\{0,0\}=0,\\ \psi(\zeta_\hbar^s(\nu,\omega))&=\psi(\zeta_\hbar^2(\nu,\omega)),\\ &=\psi\left(\sigma_1\left(\frac{p(\nu,\hbar\nu)p(\omega,\hbar\omega)}{p(\nu,\omega)}\right)^2+\sigma_2(p(\nu,\omega))^2\right)^{\frac{1}{2}},\end{aligned}$$

$$\psi(\zeta_\hbar^s(\nu,\omega))=\frac{2}{3}\left(\left(\frac{\omega^2+3\nu^2}{4}\right)^{\frac{1}{2}}\right),\tag{32}$$

$$alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega)=\left(1+\frac{\nu}{6}+\frac{\nu}{6}\right)\left(1+\frac{\omega}{6}+\frac{\omega}{6}\right)(0)=0.\tag{33}$$

From Equations (32) and (33), we attain

$$\alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega)\leq\psi(\zeta_\hbar^s(\nu,\omega)),\forall\nu,\omega\in\left[0,\frac{1}{2}\right).$$

Case (II): If $\nu,\omega\in\left[\frac{1}{2},1\right]$, then

$$\begin{aligned}p(\nu,\omega)&=\max\{\nu,\omega\}=\nu,\\ p(\nu,\hbar\nu)&=\max\{\nu,\hbar\nu\}=\max\left\{\nu,\frac{1}{6}\right\}=\nu,\\ p(\omega,\hbar\omega)&=\max\{\omega,\hbar\omega\}=\max\left\{\omega,\frac{1}{6}\right\}=\omega,\end{aligned}$$

$$\begin{aligned}p(\hbar\nu,\hbar\omega)&=\max\{\hbar\nu,\hbar\omega\}=\max\left\{\frac{1}{6},\frac{1}{6}\right\}=\frac{1}{6},\\ \psi(\zeta_\hbar^s(\nu,\omega))&=\psi(\zeta_\hbar^2(\nu,\omega)),\\ &=\psi\left(\sigma_1\left(\frac{p(\nu,\hbar\nu)p(\omega,\hbar\omega)}{p(\nu,\omega)}\right)^2+\sigma_2(p(\nu,\omega))^2\right)^{\frac{1}{2}},\end{aligned}$$

$$\psi(\zeta_\hbar^s(\nu,\omega))=\frac{(3\nu^2+\omega^2)^{\frac{1}{2}}}{3}\geq 0,\tag{34}$$

$$\alpha(\nu,\nu)\alpha(\omega,\omega)p(\hbar\nu,\hbar\omega)=\frac{1}{6}.\tag{35}$$

From Equations (34) and (35), we get

$$\alpha(\nu,\nu)\alpha(y,y)p(\hbar\nu,\hbar y) \le \psi(\zeta^{s}_{\hbar}(\nu,\omega)), \forall \nu, v \in \left[\frac{1}{2},1\right].$$

Therefore, $\hbar$ is GMKKR$(\psi-\alpha)$C.

Thus, each requirement of Theorem 1 is fulfilled. Thus, $\hbar$ has a UFP 0. However, since function α is discontinuous at $(\nu,\omega) = \left(\frac{1}{2},\frac{1}{2}\right)$, UFP cannot be determined for FPRs in [19,21]. □

Example 3. *Consider a complete PMS (χ,p) with $\chi = C[a,b]$ and $p(\nu,\omega) = \max\{\nu,\omega\}$. Define a SM $\hbar : \chi \to \chi$ defined in terms of VIE as*

$$\hbar\nu(t) = g(t) + \int_a^t K(t,s,\nu(s))ds, \forall \nu(t),\omega(t) \in \chi, t \in [a,b].$$

Then, $\hbar$ has a UFP for $s = 1, \alpha(\nu,\omega) = 1+\nu-\omega, \psi(t) = \frac{2t}{3}$ and $\sigma_1 = \sigma_2 = \frac{1}{2}$, if

1. $K(t,s,\omega(s)) \le K(t,s,\nu(s))$,
2. $g(t) + \int_a^t K(t,s,\nu(s))ds \le \frac{\nu(t)}{3}$.

For instance, $\hbar$ has UFP 0 for $\chi = C[0,1], \nu(t) = t+t^2, \omega(t) = t+t^3, g(t) = \frac{t+t^3}{6}$ and $K(t,s,\nu(s)) = \frac{s\nu(s)}{6}$.

Proof. To show the existence of UFP, it is sufficient to show that all the above conditions are satisfied. For each $\nu(t),\omega(t) \in \chi, t \in [0,1]$, we have

$$K(t,s,\nu(s)) = \frac{s\nu(s)}{6} = \frac{s^2+s^3}{6}, \tag{36}$$

$$K(t,s,\omega(s)) = \frac{s\omega(s)}{6} = \frac{s^2+s^4}{6}, \tag{37}$$

From Equations (36) and (37), we attain

$$K(t,s,\omega(s)) \le K(t,s,\nu(s)). \tag{38}$$

$$\begin{aligned} g(t) + \int_0^t K(t,s,\nu(s))ds &= \frac{t+t^3}{6} + \int_a^t \frac{s^2+s^3}{6}ds, \\ &\le \frac{t+t^2}{3}, \\ &\le \frac{\nu(t)}{3}, \end{aligned} \tag{39}$$

$$\begin{aligned} g(t) + \int_0^t K(t,s,\omega(s))ds &= \frac{t+t^3}{6} + \int_0^t \frac{s^2+s^4}{6}ds, \\ &\le \frac{t+t^3}{3}, \\ &\le \frac{\omega(t)}{3}. \end{aligned} \tag{40}$$

Thus, from Equations (38)–(40), we can say that all the above requirements are fulfilled. Therefore, $\hbar$ has a UFP.

Verification: To verify the result, it is sufficient to show that our example satisfies each assumption of our main Theorem 1.

1. $\forall \nu_0 \in \chi, \alpha(\nu_0,\nu_0) = 1 \ge 1$,

2. $\psi(t) = \frac{2t}{3} \leq \frac{2t}{3}$,
3.

$$\alpha(\nu(t), \omega(t)) = 1 + \nu(t) - \omega(t) = 1 + t^2 + t^3 \geq 1,$$
$$\alpha(\hbar\nu(t), \hbar\omega(t)) = 1 + \hbar\nu(t) - \hbar\omega(t) = 1 + \frac{1}{6}\left(\frac{t^4}{4} - \frac{t^5}{5}\right) \geq 1,$$

which shows that $\hbar$ is α-admissible.

$$p(\nu(t), \omega(t)) = \max\{\nu(t), \omega(t)\} = \max\{t + t^2, t + t^3\} = t + t^2 = \nu(t),$$
$$p(\nu(t), \hbar\nu(t)) = \max\{\nu(t), \hbar\nu(t)\} = \max\left\{t + t^2, \frac{t + \frac{4t^3}{3} + \frac{t^4}{4}}{6}\right\} = t + t^2 = \nu(t),$$
$$p(\omega(t), \hbar\omega(t)) = \max\{\omega(t), \hbar v(t)\} = \max\left\{t + t^3, \frac{t + \frac{4t^3}{3} + \frac{t^5}{5}}{6}\right\} = t + t^3 = \omega(t),$$
$$p(\hbar\nu(t), \hbar\omega(t)) = \max\{\hbar\nu(t), \hbar\omega(t)\} = \max\left\{\frac{t + \frac{4t^3}{3} + \frac{t^4}{4}}{6}, \frac{t + \frac{4t^3}{3} + \frac{t^5}{5}}{6}\right\} = \frac{t + \frac{4t^3}{3} + \frac{t^4}{4}}{6},$$
$$= \hbar\nu(t),$$
$$\psi(\zeta^s_\hbar(\nu(t), \omega(t))) = \psi\left(\sigma_1\left(\frac{p(\nu(t), \hbar\nu(t))p(\omega(t), \hbar\omega(t))}{p(\nu(t), \omega(t))}\right)^s + \sigma_2(p(\nu(t), \omega(t)))^s\right)^{\frac{1}{s}},$$
$$= \psi\left(\frac{1}{2}\left(\frac{\nu(t)\omega(t)}{\nu(t)}\right) + \frac{1}{2}(\nu(t))\right),$$
$$\psi(\zeta^s_\hbar(\nu(t), \omega(t))) = \frac{2t + t^2 + t^3}{3}, \tag{41}$$

$$\alpha(\nu(t), \nu(t))\alpha(\omega(t), \omega(t))p(\hbar\nu(t), \hbar\omega(t)) = (1)(1)(\hbar\nu(t)) = \frac{t + \frac{4t^3}{3} + \frac{t^4}{4}}{6}. \tag{42}$$

From Equations (41) and (42), we attain

$$\alpha(\nu(t), \nu(t))\alpha(\omega(t), \omega(t))p(\hbar\nu(t), \hbar\omega(t)) \leq \psi(\zeta^s_\hbar(\nu(t), \omega(t))).$$

Therefore, $\hbar$ is GMKKR($\psi - \alpha$)C. Thus, each hypothesis of Theorem 1 is satisfied. Hence, $\hbar$ has a UFP. In this example, 0 is the UFP of $\hbar$ as $t = 0, \nu(t) = \hbar\nu(t) = \omega(t) = \hbar\omega(t) = 0$. □

Example 4. *There have been plenty of additions made to the area of solutions of fractional order differential and integral equations. BCP is used to prove uniqueness of solutions of fractional differential equations [22–24].*

Consider a complete PMS (χ, p) *with* $\chi = C[0,1]$ *and* $p(\nu(t), \omega(t)) = \max\{\nu(t), \omega(t)\}$. *Define a SM* $\hbar : \chi \to \chi$ *in terms of Caputo fractional derivative as*

$$^C D^\alpha \nu(t) = \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-\tau)^{\alpha-1} f(\tau, \nu(\tau))d\tau - \frac{2t}{2-z^2}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, \nu(\tau))d\tau \right.$$
$$\left. + \frac{2t}{2-z^2}\int_0^z \left(\int_0^\tau (\tau-\mu)^{\alpha-1} f(\mu, \nu(\mu))d\mu\right)\right].$$

Then, $\hbar$ *has a UFP for* $\alpha(\nu(t), \omega(t)) = 1 + \frac{|\nu(t) - \omega(t)|}{n}, \forall \nu(t), \omega(t) \in C[0,1], n \in N, \psi(t) = \frac{2t}{3}, \forall t \in [0,1]$,
(a) $\sigma_1 = \sigma_2 = \frac{1}{2}, s = 0$, *and*
(b) $\sigma_1 = 0, \sigma_2 = 1, s > 0$,
if, for each $\nu(t), \omega(t) \in C[0,1]$*, we have the below assumptions:*

1. $\nu(t) \geq \omega(t)$,
2. $\hbar\nu(t) \leq \frac{\nu(t)}{3}$.

For instance, $\hbar$ has a UFP 0 for $\chi = C[0,1], \nu(t) = t + \frac{t^2}{2}, \omega(t) = t - t^2 + \frac{t^3}{6}, f(t, \nu(t)) = 1 + \nu(t) - t^2, \alpha(\nu(t), \omega(t)) = 1 + \frac{|\nu(t) - \omega(t)|}{n}, n \in N, \psi(t) = \frac{2t}{3}, \alpha = \frac{5}{4}, z = \frac{1}{3}$.

Proof. Since for every $t \in [0,1]$,

$$\nu(t) = t + \frac{t^2}{2}, \tag{43}$$

$$\omega(t) = \frac{t}{2} - \frac{t^2}{4} + t^3, \tag{44}$$

$$\begin{aligned}
f(t, \nu(t)) &= 1 + \nu(t) - t^2 = 1 + t - \frac{t^2}{2}, \\
\hbar\nu(t) &= \frac{1}{\Gamma(\alpha)}\bigg[\int_0^t (t-\tau)^{\alpha-1} f(\tau, \nu(\tau)) d\tau - \frac{2t}{2 - z^2}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, \nu(\tau)) d\tau \\
&\quad + \frac{2t}{2-z^2}\int_0^z \bigg(\int_0^\tau (\tau - \mu)^{\alpha-1} f(\mu, \nu(\mu)) d\mu\bigg) d\tau\bigg], \\
&= \frac{1}{\Gamma(\frac{5}{4})}\bigg[\int_0^t t^{\frac{1}{4}}\bigg(1 - \frac{\tau}{t}\bigg)^{\frac{1}{4}}\bigg(1 + \tau - \frac{\tau^2}{2}\bigg) d\tau - \frac{18t}{17}\int_0^1 (1-\tau)^{\frac{1}{4}}\bigg(1 + \tau - \frac{\tau^2}{2}\bigg) d\tau \\
&\quad + \frac{18t}{17}\int_0^z \bigg(\int_0^\tau \tau^{\frac{1}{4}}\bigg(1 - \frac{\mu}{\tau}\bigg)^{\frac{1}{4}}\bigg(1 + \mu - \frac{\mu^2}{2}\bigg) d\mu\bigg)\bigg].
\end{aligned}$$

Substituting $\frac{\tau}{t} = \xi_1$ and $\frac{\mu}{\tau} = \xi_2$ in the above equation, we get

$$\begin{aligned}
\hbar\nu(t) &= \frac{1}{\Gamma(\frac{5}{4})}\bigg[\int_0^1 t^{\frac{1}{4}}(1-\xi_1)^{\frac{1}{4}}\bigg(1 + \tau\xi_1 - \frac{\tau^2{\xi_1}^2}{2}\bigg) t d\xi_1 - \frac{18t}{17}\int_0^1 (1-\tau)^{\frac{1}{4}}\bigg(1 + \tau - \frac{\tau^2}{2}\bigg) d\tau \\
&\quad + \frac{18t}{17}\int_0^z \bigg(\int_0^1 \tau^{\frac{1}{4}}(1-\xi_2)^{\frac{1}{4}}\bigg(1 + t\xi_2 - \frac{\tau^2{\xi_2}^2}{2}\bigg)\tau d\xi_2\bigg) d\tau\bigg], \\
\hbar\nu(t) &= -0.9712t + 0.72512t^{\frac{5}{4}} + 0.32177t^{\frac{9}{4}} - 0.099704t^{\frac{13}{4}},
\end{aligned} \tag{45}$$

$$f(t, \omega(t)) = 1 + \omega(t) - t^2 = 1 + \frac{t}{2} - \frac{5t^2}{4} + t^3,$$

$$\begin{aligned}
\hbar\omega(t) &= \frac{1}{\Gamma(\alpha)}\bigg[\int_0^t (t-\tau)^{\alpha-1} f(\tau, \omega(\tau)) d\tau - \frac{2t}{2-z^2}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, \omega(\tau)) d\tau \\
&\quad + \frac{2t}{2-z^2}\int_0^z \bigg(\int_0^\tau (\tau-\mu)^{\alpha-1} f(\mu, \omega(\mu)) d\mu\bigg) d\tau\bigg], \\
&= \frac{1}{\Gamma(\frac{5}{4})}\bigg[\int_0^t t^{\frac{1}{4}}\bigg(1 - \frac{\tau}{t}\bigg)^{\frac{1}{4}}\bigg(1 + \frac{\tau}{2} - \frac{5\tau^2}{4} + \tau^3\bigg) d\tau - \frac{18t}{17}\int_0^1 (1-\tau)^{\frac{1}{4}}\bigg(1 + \frac{\tau}{2} - \frac{5\tau^2}{4} + \tau^3\bigg) d\tau \\
&\quad + \frac{18t}{17}\int_0^z \bigg(\int_0^\tau \tau^{\frac{1}{4}}\bigg(1 - \frac{\mu}{\tau}\bigg)^{\frac{1}{4}}\bigg(1 + \frac{\mu}{2} - \frac{5\mu^2}{4} + \mu^3\bigg) d\mu\bigg) d\tau\bigg], \\
&= \frac{1}{\Gamma(\frac{5}{4})}\bigg[\int_0^1 t^{\frac{1}{4}}(1-\xi_1)^{\frac{1}{4}}\bigg(1 + \frac{t\xi_1}{2} - \frac{5t^2{\xi_1}^2}{4} + t^3{\xi_1}^3\bigg) t d\xi_1 - \frac{18t}{17}\int_0^1 (1-\tau)^{\frac{1}{4}}\bigg(1 + \frac{\tau}{2} - \frac{5\tau^2}{4} + \tau^3\bigg) d\tau \\
&\quad + \frac{18t}{17}\int_0^z \bigg(\int_0^1 \tau^{\frac{1}{4}}(1-\xi_2)^{\frac{1}{4}}\bigg(1 + \frac{\tau\xi_2}{2} - \frac{5\tau^2{\xi_2}^2}{4} + \tau^3{\xi_2}^3\bigg)\tau d\xi_2\bigg) d\tau\bigg],
\end{aligned}$$

$$\hbar\omega(t) = -0.75077t + 0.72512t^{\frac{5}{4}} + 0.1064t^{\frac{9}{4}} - 0.24744t^{\frac{13}{4}} + 0.14t^{\frac{21}{4}}. \tag{46}$$

Thus, from Equations (43)–(46), we get $\forall t \in [0,1]$,

$$\hbar\nu(t) \leq \frac{\nu(t)}{3} \text{ and } \hbar\omega(t) \leq \frac{\omega(t)}{3}.$$

Hence, every hypothesis is satisfied. Thus, $\hbar$ has UFP. Further, we can verify our result in the same way as in Example 3. In this example, clearly, 0 is UFP of $\hbar$ as at $t = 0, \nu(t) = \hbar\nu(t) = \omega(t) = \hbar\omega(t) = 0$. □

Example 5. *Consider a complete PMS* (χ, p) *with* $\chi = C[0,a]$ *and,* $p(\nu(t), \omega(t)) = \max\{\nu(t), \omega(t)\}$ *and define a SM* $\hbar : \chi \to \chi$ *in terms of FDO of Riemann-Liouville type as*

$$\hbar\nu(t) = \frac{1}{\Gamma(n+1-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha} \nu(\tau) d\tau,$$
$$= \Sigma_{k=0}^{n} \frac{\nu^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(n+1-\alpha)} \int_a^t (t-\tau)^{n-\alpha} \nu^{(n+1)}(\tau) d\tau.$$

Then, $\hbar$ *has a UFP for* $\alpha(\nu(t), \omega(t)) = 1 + \frac{|\nu(t)-\omega(t)|}{n}, \forall \nu(t), \omega(t) \in C[0,a], n \in N, \psi(t) = \frac{2t}{3}, \forall t,$
(a) $\sigma_1 = \sigma_2 = \frac{1}{2}, s = 0,$
(b) $\sigma_1 = 0, \sigma_2 = 1, s > 0,$
if, for every $\nu(t), \omega(t) \in C[0,a]$, *we have*

1. $\nu(t) \geq \omega(t)$,
2. $\hbar\nu(t) \leq \frac{\nu(t)}{3}$.

For instance, $\hbar$ has a UFP for $\chi = C[0,1], \nu(t) = t - t^3, \omega(t) = \frac{t^2}{4} - 2t^3, \alpha(\nu(t), \omega(t)) = 1 + \frac{|\nu(t)-\omega(t)|}{n}, n \in N, \psi(t) = \frac{2t}{3}, \alpha = \frac{3}{2}$.

Proof. Since for all $t \in [0,1]$,

$$\nu(t) = 1 - t^3, \nu^{(1)}(t) = -3t^2, \nu^{(2)}(t) = -6t, \tag{47}$$

$$\hbar\nu(t) = \Sigma_{k=0}^{n} \frac{\nu^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(n+1-\alpha)} \int_a^t (t-\tau)^{n-\alpha} \nu^{(n+1)}(\tau) d\tau,$$
$$= \frac{\nu^{(0)}(0)t^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)} + \frac{\nu^{(1)}(0)t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} + \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t t^{-\frac{1}{2}} \left(1 - \frac{\tau}{t}\right)^{-\frac{1}{2}} (-6\tau) d\tau.$$

By substituting $\frac{\tau}{t} = \xi$, we attain

$$\hbar\nu(t) = -6.7702 t^{\frac{3}{2}}. \tag{48}$$

$$\omega(t) = \frac{t^2}{4} - 2t^3, \omega^{(1)}(t) = \frac{t}{2} - 6t^2 \text{ and } \omega^{(2)}(t) = \frac{1}{2} - 12t, \tag{49}$$

$$\hbar\omega(t) = \Sigma_{k=0}^{1} \frac{\omega^{(k)}(0)(t-0)^{k-\frac{3}{2}}}{\Gamma\left(k-\frac{1}{2}\right)} + \frac{1}{\Gamma\left(2-\frac{3}{2}\right)} \int_0^t (t-\tau)^{1-\frac{3}{2}} \omega^{(2)}(\tau) d\tau,$$

$$\hbar\omega(t) = \frac{\omega^{(0)}(0)(t)^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)} + \frac{\omega^{(1)}(0)(t)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} + \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-\tau)^{-\frac{1}{2}} \left(\frac{1}{2} - 12\tau\right) d\tau,$$

$$\hbar\omega(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} (1 - 16t) t^{\frac{1}{2}}. \tag{50}$$

Thus, from Equations (47)–(50), we attain

$$\hbar\nu(t) \leq \frac{\nu(t)}{3} \text{ and } \hbar\omega(t) \leq \frac{\omega(t)}{3},$$

for each $t \in [0,1]$. Hence, $\hbar$ has a UFP. Additionally, we can also verify our result in the same manner as in Example 3. Here, clearly, 0 is the UFP of $\hbar$ as at $t = 0, \nu(t) = \hbar\nu(t) = \omega(t) = \hbar\omega(t) = 0$. □

5. Conclusions

In this manuscript, by generalizing the idea of GMKK$(\psi - \alpha)$C, we have demonstrated the concept of GMKKR(ψ)C and GMKKR$(\psi - \alpha)$C, and established FPRs in complete PMS. Some examples are also proposed in context of our main result. Moreover, by applying our established result, solutions for VIE of second kind, FDE of Caputo type, and FDO of Riemann-Liouville type are obtained with examples in terms of FP.

In future, we may try to extend and generalize our obtained result in other spaces by further expanding the idea of GMKKR$(\psi - \alpha)$C. Additionally, we can try to attain the applications of our result in differential calculus, optimization theory, etc.

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Abbreviations

The following abbreviations are used in this manuscript:

PMS	Partial Metric Space
BCP	Banach Contraction Principle
FP	Fixed Point
FPRs	Fixed Point Results
UFP	Unique Fixed Point
SM	Self-Map
VIE	Volterra Integral Equation
FDE	Fractional Differential Equation
FDO	Fractional Differential Operator
GMKK$(\psi - \alpha)$C	Generalized Meir-Keeler-Khan Type $(\psi - \alpha)$-Contraction
GMKKR(ψ)C	Ggeneralized Meir-Keeler-Khan-Rational Type ψ-Contraction
GMKKR $(\psi - \alpha)$C	Generalized Meir-Keeler-Khan-Rational Type $(\psi - \alpha)$-Contraction

References

1. Matthews, S.G. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
2. Liouville, J. Memoire sur le developpement des fonctions ou parties de fonctions en series de sinus et de cosinus. *J. Math. Pure Appl.* **1836**, *1*, 14–32.
3. Picard, E. Memoire sur la therie des equations aux derivees partielles et la methode des approximations successives. *J. Math. Pure Appl.* **1890**, *6*, 145–210.
4. Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
5. Caccioppoli, R. Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale. *Rend. Accad. Naz. Lincei* **1930**, *11*, 357–363.
6. Jaggi, D.S. Some unique fixed point theorems. *Indian J. Pure Appl. Math.* **1977**, *8*, 223–230.
7. Karapinar, E. Revisiting the Kannan type contractions via interpolation. *Adv. Theory Nonlinear Anal. Appl.* **2018**, *2*, 85–87. [CrossRef]
8. Karapinar, E.; Fulga, A. A Hybrid contraction that involves Jaggi type. *Symmetry* **2019**, *11*, 715. [CrossRef]

9. Aydi, H.; Chen, C.M.; Karapinar, E. Interpolative Ciric-Reich-Rus type contractions via the Branciari distance. *Mathematics* **2019**, *7*, 84. [CrossRef]
10. Aydi, H.; Karapinar, E.; Roldan Lopez de Hierro, A.F. ω-interpolative Ciric-Reich-Rus-type contractions. *Mathematics* **2019**, *7*, 57. [CrossRef]
11. Mitrovic, Z.D.; Aydi, H.; Noorani, M.S.M.; Qawaqneh, H. The weight inequalities on Reich type theorem in b-metric spaces. *J. Math. Comput. Sci.* **2019**, *19*, 51–57. [CrossRef]
12. Jain, R.; Nashine, H.K.; Kadelburg, Z. Some fixed point results on relational quasi partial metric spaces and application to non-linear matrix equations. *Symmetry* **2021**, *13*, 993. [CrossRef]
13. Kumar, D.; Sadat, S.; Lee, J.R.; Park, C. Some theorems in partial metric space using auxiliary functions. *AIMS Math.* **2021**, *6*, 6734–6748. [CrossRef]
14. Saluja, G.S. Some common fixed point theorems in complete weak partial metric spaces involving auxiliary functions. *Facta Univ. FU Math. Inform.* **2022**, *37*, 951–974.
15. Keeler, E.; Meir, A. A theorem on contraction mappings. *J. Math. Anal. Appl.* **1969**, *28*, 326–329.
16. Jachymski, J. Equivalent conditions and the Meir-Keeler type theorems. *J. Math. Anal. Appl.* **1995**, *194*, 293–303. [CrossRef]
17. Samet, B. Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 4508–4517. [CrossRef]
18. Kadelburg, Z.; Radenovic, S. Meir-Keeler-type conditions in abstract metric spaces. *Appl. Math Lett.* **2011**, *24*, 1411–1414. [CrossRef]
19. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $(\alpha - \psi)$-contractive type mappings. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 2154–2165. [CrossRef]
20. Redjel, N.; Dehici, A.; Karapinar, E.; ERHAN, I. Fixed point theorems for $(\alpha - \psi)$-Meir-Keeler-khan. *J. Nonlinear Sci. Appl.* **2015**, *8*, 955–964. [CrossRef]
21. Araci, S.; Kumar, M. $(\psi - \alpha)$-Meir-Keeler-Khan type fixed point theorem in partial metric spaces. *Bol. Soc. Parana. Mat.* **2018**, *36*, 149–157.
22. Li, C.; Srivastava, H.M. Uniqueness of solutions of the generalized abel integral equations in Banach spaces. *Fractal Fract* **2021**, *5*, 105. [CrossRef]
23. Area, I.; Nieto, J.J. Fractional-order logistic differential equation with Mittag-Leffler-type kernel. *Fractal Fract* **2021**, *5*, 273. [CrossRef]
24. Burqan, A. A novel scheme of the ara transform for solving systems of partial fractional differential equations. *Fractal Fract* **2023**, *7*, 306. [CrossRef]