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CHARACTERIZATION OF δ-HUREWICZ SPACES AND SPACES OF δ-CONTINUOUS FUNCTIONS

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Abstract. This work explores the properties of δ-Hurewicz spaces, providing characterization for union of spaces, subspaces, and products of such spaces, as well as we identify under which conditions δ-continuous functions have as image a δ-Hurewicz space. These results are applied to characterize spaces of δ-Hurewicz functions. Specifically, we establish that the countable product of copies of a topological space X with product topology is δ-Hurewicz for a space X being T_1 and δ-compact and also X is δ-compact under certain hypotheses on the function space $C_\delta(X, X)$.

Keywords. δ-continuous functions, δ-Hurewicz, δ-open, topology
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1. Introduction and preliminaries

The study of covers of sets with certain properties has contributed to a considerable amount of literature on the subject. Specifically, Hurewicz spaces introduced in [4] have been recently investigated in [9] and [5], in their different versions depending on the type of sets considered in the covers. In connection with these ideas, the objective of this work is to analyze what happens when covers are formed by δ-open sets. At first glance, one might think that the properties would be preserved. However, in a study for α-open, it was observed that some of the properties are not preserved, and counterexamples were found in [3], as well as for weaker versions of the Hurewiczness concept ([2]). Continuity, faint continuity, and semi-continuity depend on the topology of the spaces involved, and the conditions also vary ([1]).

In this article, a modification of the δ-continuity introduced in [7] has been made so that constant functions are always δ-continuous.

Furthermore, for general notions of topology and continuous functions, standard notation is used. For background we refer the reader to [10] and [6].

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Let $(X, \mathcal{T})$ be a topological space. A set $D \subseteq X$ is δ-open if D is a union of regular open sets and $\mathcal{R}$ is a regular open set if $\mathcal{R} = int\,(cl\,(\mathcal{R}))$. Thus,

$$\mathcal{T}^\delta = \{D \subseteq X : D \text{ is } \delta - open\} \subseteq \mathcal{T}$$

If $Y \subseteq X$ then $\mathcal{T}_Y = \{U \bigcap Y : U \in \mathcal{T}\}$ and $(\mathcal{T}_Y)^\delta \subseteq \mathcal{T}_Y$. Note that $\mathcal{T}^\delta$ is a topology on X, since both $\emptyset$ and X are δ-open sets, arbitrary unions of δ-open sets are δ-open, and finite intersections of δ-open sets are δ-open, by the properties of regular open sets (see [10], Chapter 2).

$\mathcal{T}^\delta$ can be strictly coarser than $\mathcal{T}$. For instance, take $(\mathbb{R}, \tau_{cofinite})$,

$$\tau_{cofinite}{}^\delta = \{\mathbb{R}, \emptyset\} \neq \tau_{cofinite} = \{U \subseteq \mathbb{R} : \mathbb{R} \setminus U \text{ is finite}\} \bigcup \{\emptyset\}.$$

In this paper, $\mathbb{N} = \{1, 2, 3, \cdots\}$, the natural numbers set without the number zero.

2. δ-Hurewicz spaces

Definition 2.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A} \subseteq X$. Then $\mathcal{A}$ has the δ-Hurewicz property, if for any sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of δ-open covers of $\mathcal{A}$, there exists a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ such that $\mathcal{A} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ and for each $x \in \mathcal{A}$ there exists x belongs to all but finitely many sets $\cup\mathcal{V}_n$. A topological space $(X, \mathcal{T})$ is δ-Hurewicz space when the set X is δ-Hurewicz.

Notice that $x \in \mathcal{V}_n$ in the definition means that there exists $\mathcal{V}$ containing x in that finite collection of δ-open sets and also, for $x \in \mathcal{A}$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $x \in \mathcal{V}_n$.

Definition 2.2. A space X is said to be $\sigma - \delta$-compact if X is a countable union of δ-compact subspaces.

Proposition 2.3. *If X is $\sigma - \delta$-compact then X is δ-Hurewicz.*

Proof. Assume $X = \bigcup_{n\in\mathbb{N}} K_n$, with $K_n \subseteq K_{n+1}$, K_n δ-compact since X is $\sigma - \delta$-compact. Let $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ be a sequence of δ-open covers of X. For each $n \in \mathbb{N}$, choose $\mathcal{V}_n$ a finite subset of $\mathfrak{U}_n$ covering K_n. As $K_n \subseteq K_{n+1}$, for each $x \in X$ there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $x \in K_n$, so $x \in \mathcal{V}_n$. □

Remark 2.4. If (X, τ_{dis}) is infinity countable then it is δ-Hurewicz since X is $\sigma - \delta$-compact. Therefore, $(\mathbb{N},\ \tau_{dis})$ is $\sigma - \delta$-compact and so δ-Hurewicz.

Proposition 2.5. *If X is a finite union of δ-Hurewicz spaces, then X is δ-Hurewicz.*

Proof. Assume $X = \bigcup_{k=1}^m X_k$ with X_k δ-Hurewicz space. Let $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ be a sequence of δ-open covers of X. For each k, $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ be a sequence of δ-open covers of X_k. Choose $\mathcal{V}_n^k$ a finite subset of $\mathfrak{U}_n$ such that $X_k \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{V}_n^k$ for each $x \in X_k$ we get M_k such that $x \in \mathcal{V}_n^k$ for all $n \geq M_k$. Consider $\mathcal{W}_n = \bigcup_{j=1}^m \mathcal{V}_n^j$, a finite subset of $\mathfrak{U}_n$, $X \subseteq \bigcup\ \mathcal{W}_n$. If $x \in X$, then $x \in X_k$ for some k, so, $x \in \mathcal{V}_n^k \subseteq \mathcal{W}_n$ for all $n \geq M_k$. □

$$\beta\text{-Hurewicz} \Rightarrow s\text{-Hurewicz} \Rightarrow \alpha\text{-Hurewicz} \Rightarrow \text{Hurewicz} \Rightarrow \delta\text{-Hurewicz}$$
$$\delta\text{-compact} \Rightarrow \delta\text{-Hurewicz} \Rightarrow \delta\text{-Lindelof}$$
$$\text{Compact} \Rightarrow \delta\text{-compact}$$
$$\sigma\text{-compact} \Rightarrow \sigma - \delta\text{-compact}$$

$(\mathbb{R}, \tau_{cofinite})$ is compact. Hence, it is Hurewicz and δ-Hurewicz.

Proposition 2.6. *Let $(X, \mathcal{T})$ be a δ-Hurewicz space and $Y \subseteq X$. If Y is a δ-closed set of X, then the subspace Y is a δ-Hurewicz space with inherited topology.*

Proof. Assume $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ is a sequence of δ-open covers of Y. So, for each $n \in \mathbb{N}$, $\mathfrak{U}_n = \{U_i : i \in I_n, U_i\ \delta\text{-open of } Y\}$ $\tau_Y^\delta = \{B \subseteq Y : B\ \delta\text{-open of } Y\} \subseteq \tau_Y = \{A \cap Y : A \text{ open of } X\}$. Then $U_i = \cup B_j^i$, where $B_j^i = A_j^i \cap Y = int_Y(cl_Y(A_j^i \cap Y))$, for some open sets A_j^i of X. Now, $int_Y(cl_Y(A_j^i \cap Y)) = int(cl(A_j^i)) \cap Y$, as $\mathcal{A}_j^i = int(cl(A_j^i))$ is a regular open set of X. Thus, for each n, $\mathfrak{U}\prime_n = \{U\prime_i : i \in I_n, U\prime_i = \cup \mathcal{A}_j^i\} \cup \{X \setminus Y\}$ is δ-open cover of X. It is followed since the δ-Hurewiczness of X the existence of $\mathcal{V}\prime_n$ a finite subset of $\mathfrak{U}\prime_n$. Thus, take the intersection with Y to get suitable $\mathcal{V}_n$ for each n. □

Definition 2.7. A function $f : X \to Y$ is said to be δ-continuous if for each $x \in X$ and each δ-open V with $f(x) \in V$, there exists an δ-open U with $x \in U$ that $f(U) \subseteq V$.

Definition 2.8. A function $f : X \to Y$ is said to be δ-homomorphism if f is bijective and f, f^{-1} are δ-continuous.

Remark 2.9. If $f : X \to X$ is a constant function $f(x) = k$, for all x, then f is δ-continuous. In addition, it is followed of definition that composition of δ-continuous functions is a δ-continuous function.

Proposition 2.10. *If $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ is a surjection δ-continuous function and X is a δ-Hurewicz space then Y is a δ-Hurewicz space.*

Proof. If $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ is a sequence of δ-open covers of Y then for each $n \in \mathbb{N}$, $\mathfrak{U}_n = \{U_i^n : i \in I_n, U_i^n \delta\text{-open of Y}\}$. For each $x \in X$, $f(x) = y \in Y$, choose $U_{i_y}^n$ such that $y \in U_{i_y}^n$ and as f is δ-continuous, get $U\prime_{i_x}^n$ δ-open of X such that $x \in U\prime_{i_x}^n$ and $f(U\prime_{i_x}^n) \subseteq U_{i_y}^n$. Then $\mathfrak{U}_n\prime = U\prime_{i_x}^n$ is a cover of X for each n. X δ-Hurewicz implies the existence of $\mathcal{V}_n\prime$ a finite subset of $\mathfrak{U}_n'$. Construct $\mathcal{V}_n$ with the corresponding $U_{i_y}^n$ to $U\prime_{i_x}^n$ of $\mathcal{V}_n\prime$. As f is surjective, $Y = \cup\mathcal{V}_n$. If $y = f(x) \in Y$, there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $x \in \mathcal{V}_n$, so $y \in f(\mathcal{V}_n) \subseteq \mathcal{V}_n'$. □

Corollary 2.11. *: If $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ is δ-continuous function and X is a δ-Hurewicz space then $F(X)$ has the δ-Hurewicz property.*

Corollary 2.12. *The δ-Hurewiczness is a topological property.*

Example 2.13. The Khalimsky line, L_K is δ-Lindelof. The Khalimsky line, L_K is the set of integers $\mathbb{Z}$, equipped with the topology generated by the sets $\{2n-1, 2n, 2n+1\}$. For each integer n, $\{2n-1\}$ is open, $\{2n\}$ is closed, and $\{2n, 2n+1, 2n+2\}$ is closed. So, $\{2n-1, 2n, 2n+1\}$ is a regular set for each n.

If $\mathfrak{U} = \{U_i\}$ is a cover of L_K by δ-open set then for each $n \in \mathbb{Z}$ we choose an open set U_n of $\mathfrak{U}$ such that $n \in U_n$. Thus L_K is δ-Lindelof. In addition, the Khalimsky line is also δ-Hurewicz. To show that L_K is δ-Hurewicz, first, for each $m \in \mathbb{N}$ consider the sets:

$$A_m = \{\{-2m-1+2k, -2m+2k, -2m+1+2k\} : k = 0, \cdots, 2m\},$$

A_m is a finite collection of regular open sets, $A_m \subseteq A_{m+1}$, and $\{A_m\}_{m\in\mathbb{N}}$ covers L_K.

Suppose $\{\mathfrak{U}_m\}_{m\in\mathbb{N}}$ is a sequence of covers of L_K by δ-open sets where $\mathfrak{U}_m = U_{i i\in I_m}$. For each $m \in \mathbb{N}$ and for each $k = 0, \cdots, 2m$ choose $U_{i_{-2m+2k}}$ of $\mathfrak{U}_m$ such that $-2m+2k \in U_{i_{-2m+2k}}$ which is open, so $\{-2m+2k-1, -2m+2k, -2m+2k+1\} \subseteq U_{i_{-2m+2k}}$ Therefore, $\mathcal{V}_m = \{U_{i_{-2m+2k}} : k = 0, \cdots, 2m\}$ is a finite collection of δ-open of $\mathfrak{U}_m$, $A_m \subseteq \mathcal{V}_m$, $\bigcup_{m\in\mathbb{N}} \mathcal{V}_m = L_K$. As $A_m \subseteq A_{m+1}$, each $x \in L_K$ belongs to $\mathcal{V}_m$ for all m such that $-2m-1 \leq x \leq 2m+1$.

Example 2.14. The Michael line, L_M is not Hurewicz, not Lindelof, not δ-Lindelof and not δ-Hurewicz. The Michael line, L_M, that is, the real numbers $\mathbb{R}$ with the topology

$$\mathcal{T}_M = \{U \cup F : U \text{ is open with usual topology and } F \subseteq \mathbb{R} \setminus \mathbb{Q}\}$$

is not a Hurewicz space because it is not Lindelof since the set of irrational numbers is an uncountable open and closed and discrete set in L_M. As $\{x\}$ is a δ-open set for each irrational number and $]x, x+1/n[$ is also a δ-open set, so L_M is not a δ-Lindelof space and hence, it is not δ-Hurewicz space.

As the following example demonstrates, a subspace of a product of topological spaces need not be δ-Hurewicz.

Example 2.15. The Sorgenfrey line, L_S is neither δ-Hurewicz and nor Hurewicz, but it is δ-Lindelof and Lindelof. A subspace of the product $L_s \times L_s$ is neither δ-Hurewicz and nor Hurewicz because it is neither δ-Lindelof and nor Lindelof. This subspace can be any line L with negative slope because the inherited topology in this line is $\tau_L = \tau_{dis}$.

3. Almost δ-Hurewicz spaces, continuity and applications

The definition of the almost Hurewicz property is extended to the case of δ-Hurewiczness below.

Definition 3.1. Let X be a topological space and $\mathcal{A} \subseteq X$. $\mathcal{A}$ has the almost δ-Hurewicz property, if for any sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of δ-open cover of $\mathcal{A}$, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that $\mathcal{V}_n$ is a finite collection of δ-open sets of $\mathfrak{U}_n$, and for each $x \in \mathcal{A}$, there is $N \in \mathbb{N}$; for all $n \in \mathbb{N}$, $n \geq N$ implies that there is $\mathcal{V} \in \mathcal{V}_n$ with, $x \in cl(\mathcal{V})$. A space X is called an almost δ-Hurewicz space, if it is fulfilled the almost δ-Hurewicz property.

δ-Hurewicz ⇒ almost δ-Hurewicz

Proposition 3.2. *Almost Hurewiczness of a topological space X is equivalent to almost δ-Hurewiczness.*

Proof. (⇒) It is followed from definition and from each δ-open is open. (⇐) Let $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ be a sequence of open cover of X. Thus, $\mathfrak{U}_n = U_i$.As U_i is open then $U_i \subseteq int(cl(U_i))$. Take $W_i = int(cl(U_i))$, a regular open set, and so, it is δ-open. Hence $(\mathfrak{W}_n)_{n\in\mathbb{N}}$, $\mathfrak{W}_n = W_i$ is a sequence of δ-open cover of X. As X is almost δ-Hurewicz, then there is a sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ such that $\mathcal{W}_n$ is a finite collection of open sets of $\mathfrak{W}_n$, and for each $x \in X$, there is $N \in \mathbb{N}$; for all $n \in \mathbb{N}$, $n \geq N$ implies that there is $\mathcal{W} \in \mathcal{W}_n$ with, $x \in cl(\mathcal{W})$. Each $\mathcal{W} = int(cl(\mathcal{V}))$, $\mathcal{V}$ open set of $\mathfrak{U}_n$ Now, for each n we can choose $\mathcal{V}_n$ the finite collection of δ-open set $\mathcal{V}$ such that $\mathcal{W} = int(cl(\mathcal{V}))$ and as $cl(\mathcal{V}) = cl(int(cl(\mathcal{V})))$ then X is almost Hurewicz. □

Definition 3.3. A function $f : X \to Y$ is said to be almost δ-continuous if for each $x \in X$ and each regular open V of $f(x)$, there exists an δ-open U of x that $f(U) \subseteq V$.

Definition 3.4. A topological space X is said to be δ-regular space when for each δ-open set U of X and for each $x \in U$ there exist a δ-open set, V, such that: $x \in V \subseteq cl(V) \subseteq U$.

Proposition 3.5. *If X is δ-regular space and almost δ-Hurewicz space, then X is δ-Hurewicz space.*

Proof. If $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ is a sequence of δ-open covers of X, $\mathfrak{U}_n = \{U_i\}_{i\in I_n}$,then $\mathcal{W}_n = \{cl(W) : W \subseteq cl(W) \subseteq U_i, i \in I_n\}$ are obtained from δ-regularness. Apply δ-Hurewiczness over $\{\mathcal{W}_n\}_{n\in\mathbb{N}}$. □

Proposition 3.6. *f is δ-continuous function if and only if f is almost δ-continuous function.*

Proof. (⇒)It is obtained since each regular open V is a δ-open set.

(⇐) Let $f : X \to Y$ almost δ-continuous function. If $x \in X$ and V is δ-open with $f(x) \in V$, then $V = \cup A_i$ where A_i is regular open. Then $f(x)$ is in some A_i. From f is almost δ-continuous function there exists an δ-open U with $x \in U$ that $f(U) \subseteq A_i \subseteq V$. □

Proposition 3.7. *The image of an almost δ-Hurewicz space by an almost δ-continuous function has the almost δ-Hurewicz property.*

Proof. Assume that $f : X \to Y$, almost δ-continuous and X almost δ-Hurewicz space. Hence, f is δ-continuous. Let $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ be a sequence of δ-open covers of $f(X)$. For each $n \in \mathbb{N}$, $\mathfrak{U}_n = \{U_i^n : i \in I_n, U_i^n$ δ-open of $Y\}$. For each $x \in X$, $f(x) = y \in f(X)$, choose $U_{i_y}^n$ such that $y \in U_{i_y}^n$ and as f is δ-continuous, get $U\prime_{i_x}^n$ δ-open of X such that $x \in U\prime_{i_x}^n$ and $f(U\prime_{i_x}^n) \subseteq U_{i_y}^n$. Then $\mathfrak{U}_n\prime = U\prime_{i_x}^n$ is a cover of X for each n. X almost δ-Hurewicz implies the existence of $\mathcal{V}_n\prime$ a finite subset of $\mathfrak{U}_n\prime$. Construct $\mathcal{V}_n$ with the corresponding $U_{i_y}^n$ to $U\prime_{i_x}^n$ of $\mathcal{V}_n\prime$. If $y = f(x) \in f(X)$, we get $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $x \in cl(\mathcal{V})$, for some $\mathcal{V} \in \mathcal{V}_n$, so $y \in f(cl(\mathcal{V})) \subseteq cl(f(\mathcal{V})) \subseteq cl(\mathcal{V}\prime)$. □

Proposition 3.8. *: If $f : X \to Y$, δ-continuous function, X δ-Hurewicz, and $\mathcal{A} \subseteq X$, δ—closed, then $f(\mathcal{A})$ has the δ-Hurewicz property.*

Proof. By Proposition 2.6 it is followed that $\mathcal{A}$ is δ-Hurewicz with the inherited topology and the restriction $f_{|\mathcal{A}} : \mathcal{A} \to Y$ is δ-continuous function. Let $(\mathfrak{W}_n)_{n\in\mathbb{N}}$ be a sequence of δ-open in Y covers of $\mathcal{A}$, there exists a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$. Now, apply Corollary 2.11. □

Proposition 3.9. *If $f : X \to Y$ δ-continuous function, X δ-Hurewicz then $Gr(f) = \{(x, f(x))\} \subseteq X \times Y$ has the δ-Hurewicz property.*

Proof. Let $G : X \to X \times Y$ be a map defined by $G(x) = (x, f(x))$. Let $a \in A$, for each δ-open W_a and each δ-open $V_f(a)$, as f is δ-continuous function there exits U_a δ-open such that $f(U_a) \subseteq V_f(a)$. Take $U_a \cap W_a$. Then $G(U_a \cap W_a) \subseteq W_a \times V_f(a)$. So G is δ-continuous. Now, apply Corollary 2.11. □

Corollary 3.10. *X δ-Hurewicz then $\{(x, x) : x \in X\} \subseteq X \times X$ has δ-Hurewicz property.*

Proof. It is followed that this set is the graph of identity map. □

Example 3.11. Consider the Arens-Fort space $X = \mathbb{N}\times\mathbb{N} \bigcup (0, 0)$ The topology is generated by the next sets: X, $\emptyset$, (m, n), $m, n \in \mathbb{N}$, $U_m = (0, 0) \cup \mathbb{N} \times \mathbb{N} \setminus C_m = X \setminus C_m$, where $C_m = \{(m, n) : n \in \mathbb{N}\}$, i.e., a column. X is Lindelof but it is not Hurewicz. Indeed, properties do not work for the covers $\mathfrak{U} = \{U_n : n \in \mathbb{N}\} \cup U_0$, $U_0 = \{(0, 0)\} \cup \mathbb{N} \times \mathbb{N} \setminus C_1, U_n = \{1, n\}$ However, regular open sets are X, $\emptyset$, U_n, $n \in \mathbb{N}$. Thus, X is δ-Hurewicz since the unique δ-open which covers $(0, 0)$ is X.

Example 3.12. The Dieudonné Plank space is not δ-Hurewicz since it is not Lindelof by regular open. Thus, it is not Hurewicz. Recall the Diudonné Plank space is the space: $D = ((\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega\}$, being ω_1 the first ordinal non countable with the order topology and $\omega + 1$ is the set of natural numbers with a limit point with Alexandrov topology. Regular open sets are: Discret sets of $\omega_1 \times \omega$, $]x, \omega_1] \times \{m\}$ for $x < \omega_1$, $\{x\}\times]m, \omega]$ for $m < \omega$. Take $\mathfrak{U} = \{\{]0, \omega_1] \times \{m\}\}, \{x\}\times]0, \omega] : m \in N, x \in \omega_1\}$ There is not a countable subcover.

4. δ-Hurewicz spaces of functions

In this section consider

$$C_\delta (X, Y) = \{f : X \to Y, \delta - \text{continuous}\}.$$

Suppose a topology on $C_\delta (X, Y)$ such that the evaluation

$$\Delta_{XY} :\ C_\delta (X, Y) \times X \to Y,\ \Delta_{XY}(f, x) = f(x)$$

is a δ-continuous function with the product topology on $C_\delta (X, Y) \times X$. This hypothesis means that if we regard a function $f : X \to Y$ as an element of Y^X, projection functions $\pi_x :\ Y^X \to Y$ defined by $\pi_x(f) = f(x)$ are δ-continuous.

Proposition 4.1. *: If X is an infinity countable union of δ-Hurewicz spaces, then X is δ-Hurewicz.*

Proof. If $X = \bigcup_{k=1}^{\infty} Y_k$, Y_k a δ-Hurewicz space for all k, then $X_k = \bigcup_{j=1}^{k} Y_j$ for all k, since the finite union of δ-Hurewicz spaces is δ-Hurewicz space. So $X_k \subseteq X_{k+1}$

for all k and $X = \bigcup_{k=1}^{\infty} X_k$. Let $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ be a sequence of δ-open covers of X. Place the covers, formming a matrix,

$$\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, \cdots$$
$$., \mathfrak{U}_2, \mathfrak{U}_3, \cdots$$
$$., ., \mathfrak{U}_3, \cdots$$
$$\cdots, \mathfrak{U}_k, \mathfrak{U}_{k+1}, \cdots$$
$$\cdots$$

For each k, $(\mathfrak{U}_n)_{n\geq k}$ be a sequence of δ-open covers of X_k. For each k, we get a finite collection of δ-open sets $\mathcal{V}_n^k \subseteq \mathfrak{U}_n$, $n \geq k$ such that $\cup\mathcal{V}_n^k$ covers X_k and for each $x \in X_k$ there exists N_k such that $n\geq N_k \geq k$ implies $x \in \mathcal{V}_n^k$. We have:

$$\mathfrak{U}_1, \mathfrak{U}_2, \cdots$$

covers for X_1 and a finite collection $\mathcal{V}_n^1 \subseteq \mathfrak{U}_n$, for each $n \geq 1$

$$\mathfrak{U}_2, \mathfrak{U}_3, \cdots$$

covers for X_2 and a finite collection $\mathcal{V}_n^2 \subseteq \mathfrak{U}_n$ for each $n \geq 2$

$$\cdots$$

$$\mathfrak{U}_k, \mathfrak{U}_{k+1}, \cdots$$

covers for X_k and a finite collection $\mathcal{V}_n^k \subseteq \mathfrak{U}_n$ for each $n \geq k$ So that for X_k we do not take into account the covers that are below the main diagonal. Take $\mathcal{W}_n = \bigcup_{k=1}^{n} \mathcal{V}_n^k \subseteq \mathfrak{U}_n$, is a finite collection and $\{\cup\mathcal{W}_n\}_n$ covers X. If $x \in X$ then there is k such that $x \in X_k$ and for this k it is possible to choose N_k such that $n\geq N_k \geq k$ implies $x \in \mathcal{V}_n^k \subseteq \mathcal{W}_n$. □

Proposition 4.2. *If X is δ-Hurewicz and K is δ-compact then $X \times K$ is δ-Hurewicz with product topology.*

Proof. In general if $\{\mathfrak{W}_n\}_{n\in\mathbb{N}}$ is a sequence of covers of $X \times K$. $\mathfrak{W}_n = W_i$. Let $x_0 \in W_i$, for each k such that (x_0, k) is in W_i, there exists U_{x_0}, V_k and such that $U_{x_0} \times V_k \subseteq W_i$. For that reason, it is sufficient to prove for $(\mathfrak{W}_n)_{n\in\mathbb{N}}$, $\mathfrak{W}_n = (\mathfrak{U}_n \times \mathfrak{V}_n$, a sequence of δ-open covers of $X \times K$, $\mathfrak{U}_n$ a cover of X and $\mathfrak{V}_n$ a cover of K for each n. Then, as K is δ-compact then there is a finite subcover $\mathcal{B}_n$ covering K for each n. On the other hand, $\mathfrak{U}_n$ is a cover of X, as X is δ-Hurewicz then there is a finite collection $\mathcal{A}_n \subseteq \mathfrak{U}_n$.For each $x \in X$ there exists N such that $x \geq N$ implies $x \in \mathcal{A}_n$ Take $\mathcal{V}_n = \mathcal{A}_n \times \mathcal{B}_n$ for each n. For each $(x, k) \in X \times K$ there exists N such that $x \geq N$ implies $x \in \mathcal{A}_n$, $k \in K$, so $k \in \mathcal{B}_n$. □

Corollary 4.3. *If X is δ-Hurewicz and Y is $\sigma - \delta$-compact then $X \times Y$ is δ-Hurewicz with product topology.*

Proof. From Y is $\sigma - \delta$-compact then $Y = \bigcup_{k=1}^{\infty} Y_k$, Y_k compact, then $X \times Y = \bigcup_{k=1}^{\infty} X \times Y_k$. At this point, apply Proposition 4.1 and 4.2. □

Lemma 4.4. *If $f : X \to Y$ is δ-continuous function, K is δ-compact then $f(K)$ is δ-compact.*

Proof. Let $\mathfrak{V}$ be a cover of $f(K)$ of δ-open set of Y. Let $k \in K$, $f(k)$. Choose V of $\mathfrak{V}$ with $f(k) \in f(K)$. From $f : X \to Y$ is δ-continuous function, it is possible to find U δ-open such than $k \in U$ and $f(U) \subseteq V$. The collection of δ-open U is a cover $\mathfrak{U}$ of X. Now, as K is δ-compact, there exists a finite subcover $\{U_j\}_{j=1}^n$. Thus, its corresponding $V_{j\,j=1}^{\,n}$ covers $f(K)$. □

Proposition 4.5. *If $C_\delta(X, Y)$ endowed with a topology for which Δ_{XY} is δ-continuous and δ-Hurewicz, and X is $\sigma - \delta$-compact then Y is δ-Hurewicz.*

Proof. It is followed by constant functions are δ-continuous then Δ_{XY} is surjective and Proposition 2.10 and Lemma 4.4. □

Corollary 4.6. *If $f : X \to Y$ is surjective and δ-continuous function, X $\sigma - \delta$-compact then Y is $\sigma - \delta$-compact.*

Remark 4.7. If $\{(X_i, \tau_i)\}_{i\in I}$ is a family of topological spaces, $X_i \neq \emptyset$ for each $i \in I$, $\prod_{i\in I} X_i$ can be equipped with following topologies: $\tau_1 = \{\prod_{i\in I} U_i : U_i = X_i$ for almost everything $i \in I, U_i$ a $\delta-$open of X_i in other case$\}$ $\tau_2 = \{\prod_{i\in I} U_i : U_i = X_i$ for almost everything $i \in I, U_i$ a -open of X_i in other case$\}$, that is the product topology and then we can consider the family of δ-open, τ_2^δ. τ_1 is the product topology for (X_i, τ_i^δ). Both topologies τ_1 and τ_2^δ are equal because of next properties: 1) If $A_i \subseteq X_i$, for all $i \in I$ then $A = \prod_{i\in I} A_i$ is closed in the product topology if and only if $A_i = \emptyset$ or A_i is closed in X_i. 2)If $A_i \subseteq X_i$, for all $i \in I$ then $A = \prod_{i\in I} A_i$ is open if and only if $A = \emptyset$ or $J = \{i \in I : A_i \neq X_i\}$ is a finite set and also A_i is open in X_i, for each $i \in I$.

Theorem 4.8. *Let X be T_1. X is δ-compact if and only if $X^{\mathbb{N}}$ is δ-Hurewicz, with product topology.*

Proof. If X is δ-compact then $X^{\mathbb{N}}$ is δ-compact by Tychonoff theorem [8] applied on (X, τ^δ) and from Remark 4.7. Hence, $X^{\mathbb{N}}$ is δ-Hurewicz. Now, assume $X^{\mathbb{N}}$ is δ-Hurewicz with product topology so it is like product of copies of (X, τ^δ). On the one hand, $X^{\mathbb{N}}$ is δ-Lindelof and then X is δ-Lindelof (Section 16, [10]). On the other hand, X is countable compact if and only if each sequence of X has a frequency point. If $A \subseteq X$ is an infinite countable set without frequency points so A has not w-accumulation points. As X is T_1, A has not accumulation points. Then for each $x_n \in A$ there exists some δ-open U_n such that $(U_n \setminus x_n) \cap A = \emptyset$, $x_n \in U_n$ Hence, A is δ-closed. Fix $a_0 \in A$ $f : A \to X^{\mathbb{N}}$, $f(a) = (a_n)_{n\in\mathbb{N}}$, $a_1 = a$, $a_n = a_0$, for $n > 1$. then $f(A)$ is δ-closed set of $X^{\mathbb{N}}$. It is followed by Proposition 2.6 that $f(A)$ is δ-Hurewicz with the inherited topology. Nevertheless, the covers $\{U_n \times X \times X \cdots\}_{n\in\mathbb{N}} \cap f(A)$ has not the properties of δ-Hurewiczness. Therefore, X is countably δ-compact, and as is also δ-Lindelof then X is δ-compact. □

Corollary 4.9. *$[0, 1]^{\mathbb{N}}$ is δ-Hurewicz with product topology.*

Corollary 4.10. *$\mathbb{N}^{\mathbb{N}}$ is not δ-Hurewicz with product topology.*

Theorem 4.11. *Let X be T_1 and $C_\delta(X, X)$ endowed of a topology for which Δ_{XX} is δ-continuous. If $C_\delta(X, X)$ is δ-Hurewicz then X is δ-compact.*

Proof. Assume $C_\delta(X,X)$ is δ-Hurewicz. The constant function $f_{x_0}: X \to X$ defined by $f_{x_0}(x) = x_0$ is δ-continuous, and the projection $\pi_{x_0}: C_\delta(X,X) \to X$, defined by $\pi_{x_0}(f) = f(x_0)$ is δ-continuous since is a composition of δ-continuous functions:

$$\pi_{x_0} = \Delta_{XX} \circ (id \times f_{x_0})$$

where

$$id \times f_{x_0}: C_\delta(X,X) \times X \to C_\delta(X,X) \times X,$$
$$id \times f_{x_0}(f,x) = (f,x_0).$$

Moreover, $\pi_{x_0}(f_{x_0}) = x_0$. Thus, it is surjective. By application of 2.10 X is δ-Hurewicz, and so, X is δ-Lindelof. If $A \subseteq X$ is an infinite countable set without frequency points so A has not w-accumulation points. As X is T_1, A has not accumulation points. Then for each $x_n \in A$ there exists some δ-open U_n such that $(U_n \setminus x_n) \cap A = \emptyset$, $x_n \in U_n$ Hence, A is δ-closed. Consider $F(A) = \{f_a : a \in A\}$. As Δ_{XX} is δ-continuous $\Delta_{XX}(cl(F(A)\times X)) \subseteq cl(\Delta_{XX}(F(A)\times X)\Delta_{XX}(f_a,x) = a$ Therefore, $\Delta_{XX}(cl(F(A)\times X)) = A = \Delta_{XX}(F(A)\times X)$. If $f \in cl(F(A)) \setminus F(A)$, then $f(x) = \Delta_{XX}(f,x) \in A$. If f is not constant then there is $x_1 \neq x_2$ such that $f(x_1) \neq f(x_2) \in A$. For $U_{f(x_1)}$ chosen from $(U_n)_{n\in\mathbb{N}}$ there exists $U_{1f} \times U_{x_1}$ such that

$$\Delta_{XX}(U_{1f} \times U_{x_1}) \subseteq U_{f(x_1)},$$

$U_{2f} \times U_{x_2}$ such that $\Delta_{XX}(U_{2f} \times U_{x_2}) \subseteq U_{f(x_2)}$ and from f is in $cl(F(A))$ we get $U_{1f} \cap U_{2f} \cap F(A) \neq \emptyset$. Then for some a in A, $a = f(x_1) = f(x_2)$. Hence $F(A)$ is δ-closed. Moreover, from continuity Δ_{XX} and properties of U_n for each n it is possible to get $U\prime_n$ such that $(U_n\prime \setminus f_{x_n}) \cap F(A) = \emptyset$, $\{x_n \in U_n\}$ $F(A)$ is infinity, countable subset of $C_\delta(X,X)$, a δ-Hurewicz space, from Proposition 2.6, $F(A)$ is δ-Hurewicz. However $U\prime_n \cap F(A)$ is a δ cover of $F(A)$ that it does not satisfy the conditions of δ-Hurewiczness a contradiction. So, X is countable compact and also it is Lindelof. It is concluded that X is δ-countable compact. □

Proposition 4.12. *If X is δ-compact then $C_\delta(X,X) = X^X$ with product topology is δ-Hurewicz.*

Proof. Apply Tychonoff Theorem [8] to (X, τ^δ) and that a δ-compact space is δ-Hurewicz. □

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