Local convergence for an improved Jarratt-type method in Banach space

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***Abstract* — We present a local convergence analysis for an improved Jarratt-type methods of order at least five to approximate a solution of a nonlinear equation in a Banach space setting. The convergence ball and error estimates are given using hypotheses up to the first Fréchet derivative in contrast to earlier studies using hypotheses up to the third Fréchet derivative. Numerical examples are also provided in this study, where the older hypotheses are not satisfied to solve equations but the new hypotheses are satisfied.**

***Keywords* — Jarratt-type methods, Newton's method, Banach space, convergence ball, local convergence.**

I. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x)=0 \tag{1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1) using mathematical modelling [11, 12, 28, 30]. Moreover, artificial intelligence and e-learning are topics of increasing interest in recent years. Other authors and people from various other areas of expertise can follow these techniques to serve a community of learners. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of Numerical Functional Analysis for finding solution x^* of equation (1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev-Halley-type methods [5, 6, 11, 12, 19-27, 29, 30, 32] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet-derivative is diagonal by blocks and inexpensive or for quadratic quations the second Fréchet-derivative is constant. Moreover, in some applications involving stiff systems, high order methods are usefull. That is why in a unified way we study the local convergenve of the improved Jarratt-type method (IJTM) defined for each $n=0,1,2,\dots$ by

$$\begin{aligned}
u_n &= x_n - F'(x_n)^{-1}F(x_n),\\
y_n &= x_n + \frac{2}{3}F(u_n - x_n),\\
J_n &= (6F'(y_n) - 2F'(x_n))^{-1}(3F'(y_n) + F'(x_n)),\\
z_n &= x_n - J_nF'(x_n)^{-1}F(x_n),\\
x_{n+1} &= z_n - (2J_n - I)F'(x_n)^{-1}F(z_n),
\end{aligned}$$

where x_0 is an initial point and I is the identity operator. If we set $H_n = F'(x_n)^{-1}(F'(y_n) - F'(x_n))$, then using some algebraic manipulation we obtain that

$$J_n = \frac{1}{2}(I + (I + \frac{3}{2}H_n)^{-1}) = I - \frac{3}{4}(I + \frac{3}{2}H_n)^{-1}H_n. \tag{2}$$

This method has been shown to be of convergence order between 5 and 6 [28, 32]. The usual conditions for the semilocal convergence of these methods are ($\mathcal{C}$):

- There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \le \beta$, $\beta > 0$;
- $\|\Gamma_0 F(x_0)\| \le \eta$, $\eta \ge 0$;
- $\|F''(x)\| \le \beta_1$ for each $x \in D$, $\beta_1 \ge 0$;
- $\|F'''(x)\| \le \beta_2$ for each $x \in D$, $\beta_2 \ge 0$

or

- $\|F'''(x_0)\| \le \overline{\beta}_2$ for each $x \in D$, $\overline{\beta}_2 \ge 0$ and some $x_0 \in D$;
- $\|F'''(x) - F'''(y)\| \le \beta_3 \|x-y\|$ for each $x, y \in D$
- $\|F'''(x) - F'''(y)\| \le \varphi(\|x-y\|)$ for each $x, y \in D$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a non-decreasing function.

The local convergence conditions are similar but x_0 is x^* in ($\mathcal{C}_1$) and ($\mathcal{C}_2$). There is a plethora of local and semilocal convergence results under the ($\mathcal{C}$) conditions [1-31]. These conditions restrict the applicability of these methods. That is why, in our study we assume the conditions ($\mathcal{A}$):

- $F : D \to Y$ is Fréchet-differentiable and there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$;
- $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0 \|x - x^*\|$ for each $x \in D$;
- $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L\|x-y\|$ for each $x, y \in D$;

and

DOI: 10.9781/ijimai.2015.344

- $\| F'(x^*)^{-1}F'(x)\| \leq K$ for each $x \in D$, $k > 0$.

Notice that the ($\mathcal{A}$) conditions are weaker than the ($\mathcal{C}$) conditions. Hence, the applicability of (IJTM) is expanded under the ($\mathcal{A}$) conditions.

As a motivational example, let us define function f on $D = \overline{U}(1, \frac{3}{2})$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$f''(x) = 6x \ln x^2 + 20x^3 + 12x^2 + 10x$$

and

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that $f'''(x)$ is bounded on D. That is condition $(\mathcal{C}_4)$ is not satisfied. Hence, the results depending on $(\mathcal{C}_4)$ cannot apply in this case. However, we have $f'(x^*) = 3$ and $f(1) = 0$. That is, conditions $(\mathcal{A}_1)$ is satisfied. Moreover, conditions $(\mathcal{A}_2)$ $(\mathcal{A}_3)$ are satisfied for $L_0 = L = 146.6629073\ldots$ and $K = 101.5578008\ldots$. Then, condition $(\mathcal{A}_4)$ is also satisfied. Hence, the results of our Theorem 2.1 that follows can apply to solve equation $f(x) = 0$ using IJTM. Hence, the applicability of IJTM is expanded under the conditions $(\mathcal{A})$.

The paper is organized as follows: In Section 2 we present the local convergence of these methods. The numerical examples are given in the concluding Section 3.

In the rest of this study, $U(w,q)$ and $\overline{U}(w,q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

II. Local convergence

In this section we present the local convergence of IJTM under the $(\mathcal{A})$ conditions. It is convenient for the local convergence of IJTM to introduce some funcitons and parameters.

Let $L_0 > 0$ $L > 0$ and $K > 0$ be given constants. Define parameters r_A and r_0 by

$$r_A = \frac{2}{2L_0 + L} \tag{3}$$

and

$$r_0 = \frac{\sqrt{2}}{\sqrt{2}L_0 + L}. \tag{4}$$

Notice that

$$r_0 < r_A < \frac{1}{L_0}. \tag{5}$$

Define functions f_1 and f_2 on the interval $[0, \frac{1}{L_0})$ by

$$f_1(t) = \frac{L}{2(1 - L_0 t)} \tag{6}$$

and

$$f_2(t) = \frac{1}{3}(1 + \frac{Lt}{1 - L_0 t}). \tag{7}$$

Then, we have by the choice of r_A that

$$f_1(t) \leq 1 \quad \textit{foreach} \quad t \in [0, r_A] \tag{8}$$

and

$$f_2(t) \leq 1 \quad \textit{foreach} \quad t \in [0, r_A] \tag{9}$$

Define function f_3 on the interval $[0, \frac{1}{L_0})$ by

$$f_3(t) = \frac{(L\ )^2}{2(1 - L_0 t)^2}. \tag{10}$$

Then, we have that

$$f_3(t) \leq 1 \quad \textit{foreach} \quad t \in [0, r_0] \tag{11}$$

and

$$f_3(t) < 1 \quad \textit{foreach} \quad t \in [0, r_0) \tag{12}$$

Moreover, define functions f_4 and f_5 on the interval $[0, r_0)$ by

$$f_4(t) = \frac{Lt^2}{2(1 - L_0 t)}[1 + \frac{L^2 K t}{2(1 - L_0 t)^2 - L^2 t^2}] \tag{13}$$

and

$$f_5(t) = [1 + \frac{2K}{2(1 - L_0 t)^2 - L^2 t^2}] f_4(t). \tag{14}$$

Furthermore, define functions $\overline{f}_4$ and $\overline{f}_5$ on the interval $[0, r_0)$ by

$$\overline{f}_4(t) = f_4(t) - 1 \tag{15}$$

and

$$\overline{f}_5(t) = f_5(t) - 1 \tag{16}$$

We have that $\overline{f}_4(0) = \overline{f}_5(0) = -1 < 0$ and $\overline{f}_4(t) \to +\infty$, $\overline{f}_5(t) \to +\infty$ as $t \to r_0$. It follows by intermediate value theorem that functions $\overline{f}_4$ and $\overline{f}_5$ has zeros in $(0, r_0)$. Denote by r_4 and r_5 the minimal zeros of functions $\overline{f}_4$ and $\overline{f}_5$ on the interval $(0, r_0)$, respectively. Finally, define

$$r = \min\{r_4, r_5\} \tag{17}$$

Then, we have by the choice of r that

$$f_1(t) < 1 \tag{18}$$

$$f_2(t) < 1 \tag{19}$$

$$f_3(t) < 1 \tag{20}$$

$$f_4(t) < 1 \tag{21}$$

and

$$f_5(t) < 1 \quad foreach \quad t \in [0, r) \tag{22}$$

Next, we present the main local convergence for IJTM under the $(\mathcal{A})$ conditions.

Theorem 2.1 Suppose that the $(\mathcal{A})$ conditions and $\overline{U}(x^*, r) \subseteq D$, where r is given by (17). Then, sequence $\{x_n\}$ generated by IJTM for any $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0,1,2,\ldots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0,1,2,\ldots$

$$\| x_{n+1} - x^* \| \le f_5(\| x_n - x^* \|) \| x_n - x^* \| < \| x_n - x^* \| < r, \tag{23}$$

where function f_5 is defined by (14).

Proof. We shall use induction to show that estimates (22) hold for each $n = 0,1,2,\ldots$ Using $(\mathcal{A}_2)$ and the hypothesis $x_0 \in U(x^*, r)$, we have that

$$\| F'(x^*)^{-1}(F'(x_0) - F'(x^*)) \| \le L_0 \| x_0 - x^* \| < L_0 r < 1, \tag{24}$$

by the choice of r. It follows from (24) and the Banach lemma on invertible operators that [11, 12, 27] $F'(x_0)^{-1} \in L(Y, X)$ and

$$\| F'(x_0)^{-1} F'(x^*) \| \le \frac{1}{1 - L_0 \| x_0 - x^* \|} < \frac{1}{1 - L_0 r}. \tag{25}$$

Using the first substep of IJTM for $n = 0$, $F(x^*) = 0$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, (24) and the choice of r we get that

$$\begin{aligned} u_0 - x^* &= x_0 - x^* - F'(x_0)^{-1} F(x_0) \\ &= -(F'(x_0)^{-1} F'(x^*))[F'(x^*)^{-1} \\ &\times \int_0^1 (F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\, d\theta (x_0 - x^*)], \end{aligned} \tag{26}$$

so

$$\begin{aligned} \| u_0 - x^* \| &\le \| F'(x_0)^{-1} F'(x^*) \| \\ &\times \| F'(x^*)^{-1} \int_0^1 (F'(x^* + \theta(x_0 - x^*)) - F'(x_0)) d\theta \| \| x_0 - x^* \| \\ &\le \frac{L_0 \| x_0 - x^* \|^2}{2(1 - L_0 \| x_0 - x^* \|)} \le \frac{L \| x_0 - x^* \|^2}{2(1 - L_0 \| x_0 - x^* \|)} \\ &\le f_1(r) \| x_0 - x^* \| < \| x_0 - x^* \| < r, \end{aligned} \tag{27}$$

which shows $u_0 \in U(x^*, r)$. Using the second substep of IJTM, we get by (27) and (19) that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* + \frac{2}{3}(u_0 - x_0) \\ &= x_0 - x^* + \frac{2}{3}(u_0 - x^*) + \frac{2}{3}(x^* - x_0) \\ &= \frac{1}{3}(x_0 - x^*) + \frac{2}{3}(u_0 - x^*) \end{aligned}$$

so,

$$\| y_0 - x^* \| \le \frac{1}{3} \| x_0 - x^* \| + \frac{2}{3} \| u_0 - x^* \| \le f_2(r) \| x_0 - x^* \| < r,$$

which shows that $y_0 \in U(x^*, r)$.

Next, we shall find upper bounds on $\| H_0 \|$ and $\| J_0 \|$. Using $(\mathcal{A}_1)$, (26), (20) that

$$\begin{aligned} \frac{3}{2} \| H_0 \| &\le \frac{3}{2} \| F'(x_0)^{-1} F'(x^*) \| \| F'(x^*)^{-1}(F'(y_0) - F'(x_0)) \| \\ &\le \frac{3}{2} \frac{L \| y_0 - x_0 \|}{1 - L_0 \| x_0 - x^* \|} \le \frac{3}{2} \cdot \frac{2}{3} \frac{L \| u_0 - x_0 \|}{1 - L_0 \| x_0 - x^* \|} \\ &\le \frac{L^2 \| x_0 - x^* \|^2}{2(1 - L_0 \| x_0 - x^* \|^2)} < \left(\frac{Lr}{\sqrt{2}(1 - L_0 r)}\right)^2 \\ &= (f_3(r)^{\,2} < 1 \end{aligned} \tag{28}$$

It follows from (27) and the Banach lemma on invertible operators that $(I+\frac{3}{2}H_0)^{-1}\in L(Y,X)$ and

$$\left\|(I+\frac{3}{2}H_0)^{-1}\right\|\le\frac{1}{1-\frac{L^2\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)^2}}$$

$$<\frac{1}{1-\frac{L^2r^2}{2(1-L_0r)^2}}. \tag{29}$$

It then follows from the definition of J_0, (28) and (29) that

$$\|J_0\|\le 1+\frac{3}{4}\frac{\frac{L^2\|x_0-x^*\|^2}{3(1-L_0\|x_0-x^*\|)}}{1-\frac{L^2\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)^2}}$$

$$=1+\frac{1}{2}\cdot\frac{(1-L_0\|x_0-x^*\|)L^2\|x_0-x^*\|^2}{[2(1-L_0\|x_0-x^*\|)^2-L^2\|x_0-x^*\|^2]}. \tag{30}$$

Then, from the fourth substep of IJTM for $n=0$, (27), (28), (29), (21) and $(\mathcal{A}_4)$

$$z_0=x_0-F'(x_0)^{-1}F(x_0)+\frac{3}{4}(I+\frac{3}{2}H_0)^{-1}H_0F'(x_0)^{-1}F(x_0)$$

so,

$$\|z_0-x^*\|\le\|x_0-x^*-F'(x_0)^{-1}F(x_0)\|$$

$$+\frac{3}{4}\left\|(I+\frac{3}{2}H_0)^{-1}\right\|\|H_0\|\|F'(x_0)^{-1}F'(x^*)\|$$

$$\times\left\|F'(x^*)^{-1}\int_0^1F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta\right\|$$

$$\le\frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}+\frac{3}{4}\frac{1}{1-\frac{L^2\|x_0-x^*\|}{2(1-L_0\|x_0-x^*\|)}}$$

$$=\frac{2}{3}\frac{L^2\|x_0-x^*\|}{2(1-L_0\|x_0-x^*\|)^2}\frac{K\|x_0-x^*\|}{1-L_0\|x_0-x^*\|)}$$

$$=f_4(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r, \tag{31}$$

which shows $z_0\in U(x^*,r)$.

Notice that we used

$$F(x_0)=F(x_0)-F(x^*)=\int_0^1F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta$$

so

$$\|F'(x^*)^{-1}F(x_0)\|\le K\|x_0-x^*\|\quad by\quad (A_4). \tag{32}$$

Next, using the last substep in IJTM for $n=0$, (25), (29), (21) and (32) (for x_0 replaced by z_0) we get in turn that

$$\|x_1-x^*\|\le\|z_0-x^*\|+\frac{1}{1-\frac{L^2\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)^2}}\frac{K\|z_0-x^*\|}{1-L_0\|x_0-x^*\|}$$

$$\times[1+\frac{2K(1-L_0\|x_0-x^*\|)}{2(1-L_0\|x_0-x^*\|)^2-L^2\|x_0-x^*\|^2}]\|z_0-x^*\|$$

$$\le f_5(\|x_0-x^*\|)\|x_0-x^*\|\le f_5(r)\|x_0-x^*\|$$

$$<\|x_0-x^*\|, \tag{33}$$

which shows (23) for $n=0$.

To complete the induction, simple replace in all preceding estimates x_0,u_0,y_0,z_0,x_1 by x_k,u_k,y_k,z_k,x_{k+1}, respectively to arrive at (23), which complete the induction.

Finally it follows from (23) that $\lim_{k\to\infty}x_k=x^*$ ▪

Remark 2.2

1. Condition ($\mathcal{A}_2$) can be dropped, since this condition follows from ($\mathcal{A}_3$). Notice, however that

$$L_0\le L \tag{34}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2-6].

2. In view of condition ($\mathcal{A}_2$) and the estimate

$$\|F'(x^*)^{-1}F'(x)\|=\|F'(x^*)^{-1}[F'(x)-F'(x^*)]+I\|$$

$$\le 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\|$$

$$\le 1+L_0\|x-x^*\|,$$

condition ($\mathcal{A}_4$) can be dropped and K can be replaced by

$$K(r)=1+L_0r. \tag{35}$$

3. It is worth noticing that r is such that

$$r<r_A\ \ for\ \alpha\ne 0 \tag{36}$$

The convergence ball of radius r_A was given by us in [2, 3, 5] for Newton's method under conditions ($\mathcal{A}_1$)-($\mathcal{A}_3$). Estimate (24) shows that the convergence ball of higher than two IJTM methods is smaller than the convergence ball of the quadratically convergent Newton's method. The convergence ball given by Rheinboldt [30] for Newton's method is

$$r_R=\frac{2}{3L}<r_A \tag{37}$$

if $L_0 < L$ and $\frac{r_R}{r_A} \to \frac{1}{3}$ as $\frac{L_0}{L} \to 0$. Hence, we do not expect r to be larger than r_A no matter how we choose L_0, L and K. Finally note that if $\alpha = 0$, then IJTM reduces to Newton's method and $r = r_A$.

4. The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [11, 12, 30].

5. The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [11, 12, 28, 30]:

$$F'(x) = T(F(x)), \tag{38}$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

6. It is worth noticing that IJTM is not changing if we use the ($\mathcal{A}$) instead of the ($\mathcal{C}$) conditions. Moreover for the error bounds in practice we can use the computational order of convergence (COC) [1-4, 11, 12, 14] using

or the approximate computational order of convergence (ACOC)

$$\xi = \sup \frac{\ln\left(\frac{\| x_{n+2} - x_{n+1} \|}{\| x_{n+1} - x_n \|}\right)}{\ln\left(\frac{\| x_{n+1} - x_n \|}{\| x_n - x_{n-1} \|}\right)} \quad foreach \quad n = 1,2,\ldots$$

instead of the error bounds obtained in Theorem 2.1.

III. Numerical examples

$$\xi^* = \sup \frac{\ln\left(\frac{\| x_{n+2} - x^* \|}{\| x_{n+1} - x^* \|}\right)}{\ln\left(\frac{\| x_{n+1} - x^* \|}{\| x_n - x^* \|}\right)} \quad foreach \quad n = 0,1,2,\ldots$$

We present numerical examples where we compute the radii of the convergence balls.

Example 3.1 *Let* $X = Y = R^3$, $D = \overline{U}(0,1)$. *Define* F *on* D *for* $v = (x, y, z)$ *by*

$$F(v) = (e^x - 1, \frac{e-1}{2} y^2 + y, z). \tag{39}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = K = e$, $r_0 = 0.274695\ldots < r_A = 0.324967\ldots < 1/L_0 = 0.581977\ldots$, $r = 0.144926\ldots$.

Example 3.2 *Let* $X = Y = C([0,1])$, *the space of continuous functions defined on* $[0,1]$ *be and equipped with the max norm. Let* $D = \overline{U}(0,1)$. *Define function* F *on* D *by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{40}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2 \xi(\theta) d\theta, \ foreach\ \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$ and $K = K(t) = 1 + 7.5t$, $r_0 = 0.055228\ldots < r_A = 0.066666\ldots < 1/L_0 = 0.133333\ldots, r = 0.0370972\ldots$.

Example 3.3 *Returning to the motivational example at the Introduction of this study, let the function* f *on* $D = \overline{U} = (1, \frac{3}{2})$ *defined by*

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then, $L_0 = L = 146.662907\ldots$, $K = 101.557800\ldots$, $r_0 = 0.003984\ldots < r_A = 0.004545\ldots < 1/L_0 = 0.006818\ldots$ and $r = 0.000442389\ldots$.

IV. Acknowledgements

This scientific work has been supported by the 'Proyecto Prometeo' of the Ministry of Higher Education, Science, Technology and Innovation of the Republic of Ecuador.

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